Варіант 9.

(1)

$$\psi(\rho) = \rho^2 \exp\left[-\frac{\rho}{4}\right] \quad ; \quad \vec{f} = \frac{d^2}{d\rho^2} + \frac{1}{\rho} - \frac{2}{\rho^2}$$

$$\frac{d}{d\rho}\psi(\rho) = 2\rho \exp\left[-\frac{\rho}{4}\right] - \frac{\rho^2}{4}\exp\left[-\frac{\rho}{4}\right] = \left(2\rho - \frac{\rho^2}{4}\right)\exp\left[-\frac{\rho}{4}\right];$$

$$\frac{d^2}{d\rho^2}\psi(\rho) = \left(2 - \frac{\rho}{2}\right)\exp\left[-\frac{\rho}{4}\right] - \frac{1}{4}\left(2\rho - \frac{\rho^2}{4}\right)\exp\left[-\frac{\rho}{4}\right] =$$

$$= \exp\left[-\frac{\rho}{4}\right]\cdot\left(2 - \frac{\rho}{2} - \frac{\rho}{2} + \frac{\rho^2}{16}\right) = \exp\left[-\frac{\rho}{4}\right]\left(2 - \rho + \frac{\rho^2}{16}\right);$$

$$\vec{f}\psi(\rho) = \exp\left[-\frac{\rho}{4}\right]\left(\cancel{2-\rho} + \frac{\rho^2}{16} + \cancel{\rho - 2}\right) = \frac{\rho^2}{16}\exp\left[-\frac{\rho}{4}\right];$$

$$\boxed{\vec{f}\psi(\rho) = \frac{1}{16}\psi(\rho) \quad ; \quad \lambda = \frac{1}{16}}$$ – власне значення

↪ Доведено, що $\varphi(\rho)$ є власною функцією оператора $\vec{f}$.

(2) Нескінченно глибока прямокутна потенціальна яма ширина a. Знайти $\langle x^2 \rangle$.

Розв'язок

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \varphi^*(x)\, x^2\, \varphi(x)\, dx \quad ;$$

// Хвильова ф-ія частинки всередині ями $\varphi(x) = \sqrt{\frac{2}{a}}\, e^{i\alpha} \sin\frac{n\pi x}{a}$, поза ямою $\varphi(x) = 0$;

$$\boxed{\langle x^2 \rangle} = \int_0^a x^2 \cdot \frac{2}{a}\left(e^{-i\alpha}\cdot e^{i\alpha}\right)\cdot \sin^2\frac{n\pi x}{a}\, dx = \frac{2}{a}\int_0^a x^2 \sin^2\frac{n\pi x}{a}\, dx = \left|\; x^2 \to 2x\,dx \quad \sin^2\frac{n\pi x}{a}dx \to \right.$$

$$\to \frac{1 - \cos\left(\frac{2n\pi x}{a}\right)}{2}dx \to \frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{a}\right)}{2}\cdot\frac{a}{2n\pi} \Rightarrow$$

Второе слагаемое отрицательное!

$$\Rightarrow \frac{2}{a}\int_0^a 2x\left(\frac{x}{2} - \frac{a}{4n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right)dx = \left(\frac{2}{a}\int_0^a x^2 dx\right) - \left(\frac{1}{n\pi}\int_0^a x \sin\left(\frac{2n\pi x}{a}\right)dx\right) =$$

$$\Rightarrow \frac{2}{a}\cdot\frac{x^3}{3}\Big|_0^a - \quad // \; x \to dx \quad \sin\left(\frac{2n\pi x}{a}\right)dx \to -\cos\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi} \; // \Rightarrow$$

$$\Rightarrow \frac{2}{a}\cdot\frac{a^3}{3} - \frac{1}{n\pi}\left(x\left(-\cos\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi}\right)\Big|_0^a - \int_0^a -\cos\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi}dx\right) \Rightarrow$$

$$\Rightarrow \frac{2a^2}{3} - \frac{1}{n\pi}\left(-\frac{a^2}{2n\pi} + \frac{a}{2n\pi}\int_0^a \cos\left(\frac{2n\pi x}{a}\right)dx\right) \to \Big/ \sin\left(\frac{2n\pi x}{a}\right)\cdot\frac{a}{2n\pi}\Big|_0^a = 0 \Big/ \Rightarrow$$

$$\Rightarrow \frac{2a^2}{3} + \frac{a^2}{2n^2\pi^2} \quad // \to \frac{2}{a}\cdot x^2\left(\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{a}\right)}{2}\cdot\frac{a}{2n\pi}\right)\Big|_0^a = \frac{a^3}{2}\cdot\frac{2}{a} - 0 = a^2;$$

$$\to a^2 - \frac{2a^2}{3} + \frac{a^2}{2n^2\pi^2} = \frac{a^2}{3} + \frac{a^2}{2n^2\pi^2} = a^2\left(\frac{1}{3} + \frac{1}{2n^2\pi^2}\right)$$

③ На частинку з масою m, яка знаходиться в нескінченно глибокій одновимірній потенціальній ямі шириною a, діє збурення $U(x) = U_0\left[1 - \left|\frac{2x}{a} - 1\right|\right]$. Обчислити у першому порядку теорії збурень зсув n-го енергетичного рівня і знайти критерій застосовності результату.

Розв'язок

Нульовий порядок по λ: $c_m^{(0)} = \delta_{mk}$, $E^{(0)} = E_k$.

Перший порядок по λ: $E^{(1)} = \tilde{V}_{kk}$;

Другий порядок по λ: $E^{(2)} = \sum\limits_{n\neq k} \frac{|\tilde{V}_{nk}|^2}{(E_k - E_n)}$

$$E = E_k + \lambda \tilde{V}_{kk} + \sum_{n\neq k} \frac{\lambda^2 |\tilde{V}_{nk}|^2}{(E_k - E_n)} + \dots = E_k + V_{kk} + \sum_{n\neq k} \frac{|V_{nk}|^2}{(E_k - E_n)} + \dots$$

$$V_{mn} = \int \psi_m^*(q)\, \hat{V}(q)\, \psi_n(q)\, dq \quad ; \quad V_{mn} = \lambda \tilde{V}_{mn}$$

$\psi(x) = \sqrt{\frac{2}{a}} \cdot e^{i\alpha} \cdot \sin\left(\frac{n\pi x}{a}\right)$ – хвильова ф-ія частинки всередині потенціальної ями

$$V_{kk} = \int_0^a \frac{2}{a} \cdot \sin^2\left(\frac{n\pi x}{a}\right) \cdot U_0 \cdot \left(1 - \left|\frac{2x}{a} - 1\right|\right) dx \quad ;$$

$0 \quad - \quad \frac{a}{2} \quad + \quad a$

$\left[0, \frac{a}{2}\right]$: $\cancel{1 + \frac{2x}{a}}$ $1 + \frac{2x}{a} - 1 = \frac{2x}{a}$

$\left[\frac{a}{2}; a\right]$: $1 - \frac{2x}{a} + 1 = 2\left(1 - \frac{x}{a}\right)$

$$\boxed{V_{kk}} = \int_0^{a/2} \frac{2}{a} U_0 \cdot \frac{2x}{a} \sin^2\left(\frac{n\pi x}{a}\right) dx + \int_{a/2}^{a} \frac{2}{a} U_0 \cdot 2\left(1 - \frac{x}{a}\right) \sin^2\left(\frac{n\pi x}{a}\right) dx \Rightarrow$$

$\Rightarrow$ // $2x\,dx = dx^2$ // $\sin^2\left(\frac{n\pi x}{a}\right) = \frac{1 - \cos\left(\frac{2n\pi x}{a}\right)}{2}$;// $\Rightarrow$

$$\Rightarrow \int_0^{a/2} \frac{2U_0}{a^2} x \left(1 - \cos\left(\frac{2n\pi x}{a}\right)\right) dx + \int_{a/2}^{a} \frac{2U_0}{a}\left(1 - \frac{x}{a}\right)\left(1 - \cos\left(\frac{2n\pi x}{a}\right)\right) dx$$

$$\int_{0;\ a/2}^{a/2;\ a} x \cos\left(\frac{2n\pi x}{a}\right) dx = \left| \begin{matrix} x \to dx \\ \cos(\dots)\, dx \Rightarrow \sin(\dots)\frac{a}{2n\pi} \end{matrix} \right| = \frac{x \cdot a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right)\Bigg|_{0;\ a/2}^{a/2;\ a} \overset{0}{} -$$

$$- \int_0^{a/2} \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) dx = \frac{a^2}{4n^2\pi^2} \cos\left(\frac{2n\pi x}{a}\right)\Bigg|_0^{a/2} = \frac{a^2}{4n^2\pi^2}\left((-1)^n - 1\right)$$

$$\Big|_{a/2}^{a} = \frac{a^2}{4n^2\pi^2}\left(1 - (-1)^n\right)$$

$$\frac{2U_0}{a^2}\int_0^{a/2} x\,dx - \frac{2U_0}{a^2}\int_0^{a/2} x\cos\left(\frac{2n\pi x}{a}\right)dx + \frac{2U_0}{a}\int_{a/2}^{a} dx - \frac{2U_0}{a^2}\int_{a/2}^{a} x\,dx \;=$$

$$= \underbrace{\frac{2U_0}{a}\int_{a/2}^{a}\cos\left(\frac{2n\pi x}{a}\right)dx}_{=0} + \frac{2U_0}{a^2}\int_{a/2}^{a} x\cos\left(\frac{2n\pi x}{a}\right)dx \;=$$

$$= \frac{2U_0}{a^2}\cdot\frac{x^2}{2}\Big|_0^{a/2} - \frac{2U_0}{a^2}\frac{a^2}{4n^2\pi^2}\left((-1)^n-1\right) + \frac{2U_0}{a}x\Big|_{a/2}^{a} - \frac{2U_0}{a^2}\frac{x^2}{2}\Big|_{a/2}^{a} \;\oplus$$

$$\oplus\; \frac{2U_0}{a^2}\cdot\frac{a^2}{4n^2\pi^2}\left(1-(-1)^n\right) = \frac{2U_0}{2a^2}\cdot\frac{a^2}{4} + \frac{2U_0}{a}\cdot\frac{a}{2} - \frac{2U_0}{2a^2}\left(a^2-\frac{a^2}{4}\right) \;\oplus$$

$$\oplus\; \frac{2U_0}{a^2}\cdot\frac{a^2}{4n^2\pi^2}\left(1-(-1)^n\right)\cdot 2 = \frac{U_0}{4} + U_0 - \frac{3U_0}{4} + \frac{U_0}{n^2\pi^2}\left(1-(-1)^n\right) \;=$$

$$= \frac{U_0}{2} + \frac{U_0}{n^2\pi^2}\left(1-(-1)^n\right) \rightarrow \begin{cases} n=2k: & \dfrac{U_0}{2} \\ n=2k+1: & U_0\left(\dfrac{1}{2}+\dfrac{2}{(2k+1)^2\pi^2}\right) \end{cases}$$

Зі зростанням $k \to \infty$ монотонно прямує до значення $\frac{U_0}{2}$.

Критерій збіжності (критерій застосовності):

$V_{kk} << E_k$; $\left|\frac{V_{mk}}{(E_k-E_m)}\right| << 1$, $m \neq k$;

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n=1,2,3\ldots$$

$$\boxed{V_{kk} << E_k} \rightarrow \quad n=2k+1=1 \Rightarrow U_0\left(\frac{1}{2}+\frac{2}{\pi^2}\right) \le \frac{\pi^2\hbar^2}{2ma^2}\cdot\frac{1}{10}$$

$$\left(\frac{1}{2}+\frac{2}{\pi^2}\right) = \frac{\pi^2+4}{2\pi^2} \Rightarrow U_0 \le \frac{2\pi^2}{\pi^2+4}\cdot\frac{1}{10}\cdot\frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^4\hbar^2}{10ma^2(\pi^2+4)}$$

4) $U = C|x|$. Пробна функція $\varphi = A\exp\left(-\frac{x^2}{2\alpha}\right)$. Знайти енергію основного стану і хвильову ф-ю основного стану варіаційним методом.

Умова нормування: $1 = \int_{-\infty}^{\infty}\varphi^*\varphi\,dx = A^2\cdot\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)dx = A^2\cdot\sqrt{\alpha\pi}$.

Відповідно: $\boxed{A = \frac{1}{(\alpha\pi)^{1/4}}}$; $\hat{H} = \hat{V} + \hat{T} = C|x| - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$$V(\alpha) = \int_{-\infty}^{\infty}\varphi^*\hat{H}\varphi\,dx = A^2\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\alpha}\right)\left(C|x| - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\exp\left(-\frac{x^2}{2\alpha}\right)dx \;=$$

$$= A^2C\int_{-\infty}^{\infty}|x|\exp\left(-\frac{x^2}{\alpha}\right)dx - \frac{A^2\hbar^2}{2m}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2\alpha}\right)\frac{d^2}{dx^2}\exp\left(-\frac{x^2}{2\alpha}\right)dx \;=$$

$$= \Big/\; A^2C\int_0^{\infty}2x\exp\left(-\frac{x^2}{\alpha}\right)dx = -\alpha A^2C\int_0^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)d\left(-\frac{x^2}{\alpha}\right) = \alpha A^2C \;\Big/$$

// $\frac{d^2}{dx^2}\exp\left(-\frac{x^2}{2\alpha}\right) \to \frac{d}{dx}\exp\left(-\frac{x^2}{2\alpha}\right) = -\frac{x}{\alpha}\exp\left(-\frac{x^2}{2\alpha}\right)$

$\frac{d^2}{dx^2}\exp\left(-\frac{x^2}{2\alpha}\right) \Rightarrow -\frac{1}{\alpha}\left(\exp\left(-\frac{x^2}{2\alpha}\right) - \frac{x^2}{\alpha}\exp\left(-\frac{x^2}{2\alpha}\right)\right) =$

$= \exp\left(-\frac{x^2}{2\alpha}\right)\cdot\left(\frac{x^2}{\alpha^2} - \frac{1}{\alpha}\right)$ //

$\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)\cdot\left(\frac{x^2}{\alpha^2} - \frac{1}{\alpha}\right) = \frac{1}{\alpha^2}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{x^2}{\alpha}\right)dx - \frac{1}{\alpha}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)dx =$

$=$ / $\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)dx = \sqrt{\pi\sqrt{\alpha}} = \pi^{\frac12}\alpha^{\frac14}$; / $\frac{d}{d\alpha}\exp\left(-\frac{x^2}{\alpha}\right) = \exp\left(-\frac{x^2}{\alpha}\right)\cdot\frac{x^2}{\alpha^2}$

$\frac{d}{d\alpha}\left(\pi^{\frac12}\alpha^{\frac14}\right) = \frac14\pi^{\frac12}\alpha^{-\frac34}$; // $\int_{-\infty}^{\infty}\frac{x^2}{\alpha^2}\exp\left(-\frac{x^2}{\alpha}\right)dx = \frac14\pi^{\frac12}\alpha^{-\frac34}$;

$\int_{-\infty}^{\infty}\frac{x^2}{\alpha^2}\exp\left(-\frac{x^2}{\alpha}\right)dx - \frac{1}{\alpha}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{\alpha}\right)dx = \frac14\pi^{\frac12}\alpha^{-\frac34} - \pi^{\frac12}\alpha^{-\frac34} = \boxed{-\frac{3\alpha^{-\frac34}\pi^{\frac12}}{4}}$

$\Rightarrow \alpha A^2 C + \frac{A^2\hbar^2}{2m}\cdot\frac{3\alpha^{-\frac34}\pi^{\frac12}}{4} = J(\alpha) =$ / $A^2 = \left(\frac{1}{\alpha\pi^2}\right)^{\frac14}$ / $\Rightarrow$

$\Rightarrow \alpha\cdot\alpha^{-\frac14}\pi^{-\frac12}C + \frac{\alpha^{-\frac14}\pi^{-\frac12}\cdot\hbar^2}{2m}\cdot\frac{3\alpha^{-\frac34}\pi^{\frac12}}{4} = \alpha^{\frac34}\pi^{-\frac12}C + \frac{3\hbar^2}{8m\alpha}$ $\Rightarrow \min J(\alpha) = 0$

$\cancel{\alpha^{\frac74}\pi^{-\frac12} = -\frac{3\hbar^2}{8m} \to \alpha^{\frac74} = -\frac{3\hbar\pi^{\frac12}}{8mC} \to \alpha = \left(-\frac{3\hbar\pi^{\frac12}}{8mC}\right)^{\frac47}}$

$\cancel{\psi = \alpha^{-\frac18}\pi^{-\frac14}\exp\left(-\frac{x^2}{2}\left(\frac{3\hbar\pi^{\frac12}}{8mC}\right)^{-\frac47}\right)}$

E_0 $\frac{d}{d\alpha}\left(\alpha^{\frac34}\pi^{-\frac12}C + \frac{3\hbar^2}{8m\alpha}\right) = \frac34\alpha^{-\frac14}\pi^{-\frac12}C - \frac{3\hbar^2}{8m\alpha^2} = 0$

$\alpha^{-\frac14}\pi^{-\frac12}C = \frac{\hbar^2}{2m\alpha^2}$; $\alpha^{\frac74} = \frac{\hbar^2\pi^{\frac12}}{2mC} \to \boxed{\alpha = \left(\frac{\hbar^2\pi^{\frac12}}{2mC}\right)^{\frac47}}$

$$\boxed{\psi = \alpha^{-\frac18}\pi^{-\frac14}\exp\left(-\frac{x^2}{2}\left(\frac{2mC}{\hbar^2\pi^{1/2}}\right)^{4/7}\right)}$$

$$\boxed{E_0 = \pi^{-\frac12}C\left(\frac{\hbar^2\pi^{\frac12}}{2mC}\right)^{\frac37} + \frac{3\hbar^2}{8m}\left(\frac{2mC}{\hbar^2\pi^{1/2}}\right)^{4/7}}$$

Варіант 7.

(1) $\psi_1 = \psi_{100} = \frac{1}{\sqrt{\pi a^3}} \exp\left(-\frac{r}{a}\right)$;

$\psi_2 = \psi_{21-1} = \frac{r\sin\theta}{8a\sqrt{\pi a^3}} \exp\left(-\frac{r}{a}\right)\exp(-i\varphi)$;

$\psi = 0{,}5\psi_1 + C\psi_2$; При суперпозиції має виконуватись умова $\sum C_n^2 = 1$;

$0{,}5^2 + C^2 = 1 \to 1 - 0{,}25 = 0{,}75 = C^2$; $C = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$;

Знайти $\sqrt{\langle r^2 \rangle}$.

Розв'язок.

$\langle r^2 \rangle = \int \psi^*(r)\, \hat{r}^2 \psi(r)\, dV$; $dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$\langle r^2 \rangle \Rightarrow \boxed{I^2)}\ \frac{1}{4}\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{\pi a^3} \exp\left(-\frac{2r}{a}\right) r^2 \underbrace{\sin\theta\, dr\, d\theta\, d\varphi}_{\to 4\pi} \Rightarrow$

$\Rightarrow \int_0^\infty \frac{r^2}{a^3} \exp\left(-\frac{2r}{a}\right) dr = \Big/ \begin{matrix} r^2 \to 2r\,dr \\ \exp\left(-\frac{2r}{a}\right)dr \to -\frac{a}{2}\exp\left(-\frac{2r}{a}\right) \end{matrix} \Big/\!\!\Big/ = \cancel{\frac{a}{2}}\cancel{2} \int_0^\infty r \exp\left(-\frac{2r}{a}\right) dr =$

$= \Big/ \begin{matrix} r \to dr \\ \exp\left(-\frac{2r}{a}\right)dr \to -\frac{a}{2}\exp\left(-\frac{2r}{a}\right) \end{matrix} \Big/\!\!\Big/ \to a \cdot \frac{a}{2} \int_0^\infty \exp\left(-\frac{2r}{a}\right) dr = \frac{a^2}{2}\left(-\frac{a}{2}\right) \exp\left(-\frac{2r}{a}\right)\Big|_0^\infty = \boxed{\frac{a^3}{4}}$

$II^2 = \langle \psi_2^* \psi_2 \rangle \to \frac{3}{4} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin^2\theta}{64\pi a^5} \exp\left(-\frac{2r}{a}\right) r^2 dr\, \sin\theta\, d\theta\, \underbrace{d\varphi}_{\to 2\pi} \Rightarrow$

$\Rightarrow \frac{3}{128 a^5} \int_0^\infty \int_0^\pi \sin^3\theta\, d\theta \cdot r^4 \exp\left(-\frac{2r}{a}\right) dr = \Big/\!\!\Big/ -\sin^2\theta\, d(\cos\theta) \to (\cos^2\theta - 1)\, d\cos\theta$

$\frac{\cos^3\theta}{3}\Big|_0^\pi - \cos\theta\Big|_0^\pi = \left(\frac{-1-1}{3}\right) - (-1-1) \Rightarrow$

$\Rightarrow -\frac{2}{3} + 2 = \frac{4}{3}\ \Big/\!\!\Big/$

$\Rightarrow \Big/\!\!\Big/ \int_0^\infty r^4 \exp\left(-\frac{2r}{a}\right) dr = 4 \cdot 3 \cdot 2 \cdot 1 \left(+\frac{a}{2}\right)^4 \int_0^\infty \exp\left(-\frac{2r}{a}\right) dr = -\left(\frac{a}{2}\right)^5 \cdot 4!\, \exp\left(-\frac{2r}{a}\right)\Big|_0^\infty \Rightarrow$

$\Rightarrow \frac{a^5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}}{\cancel{4} \cdot \cancel{2} \cdot 4} = \frac{3a^5}{4} \longrightarrow \frac{3}{128 \cancel{a^5}} \cdot \cancel{\frac{4}{3}} \cdot \cancel{\frac{3a^5}{4}} = \frac{5}{128}$?!

$\int$ в интегралах (в аргументе) пропущено домножение на r^2

② Визначити комутатор операторів $\frac{d}{dx}+\frac{d}{dy}$ та xy.

Розв'язок.

$$\left[\left(\frac{d}{dx}+\frac{d}{dy}\right)(xy\psi)\right]-\left[xy\cdot\left(\frac{d\psi}{dx}+\frac{d\psi}{dy}\right)\right] =$$

$$= y\psi + xy\frac{d\psi}{dx} + x\psi + xy\frac{d\psi}{dy} - xy\frac{d\psi}{dx} - xy\frac{d\psi}{dy} =$$

$$= \psi(x+y) + \frac{d\psi}{dx}(\cancel{xy-xy}) + \frac{d\psi}{dy}(\cancel{xy-xy}) = \psi(x+y) \to$$

$$\to \boxed{\left[\widehat{\left(\frac{d}{dx}+\frac{d}{dy}\right)}, \hat{xy}\right] = x+y}$$

③ $\boxed{\text{3-тя задача з варіанта 9}}$

④ Знайти варіаційним методом Рітца наближене знач. енергії осн. стану

Пробна функція: $\psi(x) = A\left(1+\frac{x^2}{a^2}\right)^{-2}$; $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U(x)$

$U(x)=\frac{m\omega^2x^2}{2}$

Розв'язок

$$1=\int_{-\infty}^{\infty}\psi^*\psi\,dx = A^2\int_{-\infty}^{\infty}\frac{dx}{\left(1+\frac{x^2}{a^2}\right)^4} = \frac{5\pi a}{16}A^2 = 1 \to \boxed{A=\sqrt{\frac{16}{5\pi a}}}$$

$$1+\frac{x^2}{a^2}=0 \to x^2=-a^2 \to x=\begin{cases}+ia\\-ia\end{cases} \to \;\to$$

$$E = \int_{-\infty}^{+\infty}\psi^*\hat{H}\psi\,dx = \int_{-\infty}^{+\infty}A^2\left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\left(-\frac{\hbar^2}{2m}\right)\frac{d^2}{dx^2}\left(1+\frac{x^2}{a^2}\right)^{-2}dx +$$

$$+\int_{-\infty}^{\infty}A^2\left(1+\frac{x^2}{a^2}\right)^{-4}\cdot\frac{m\omega^2x^2}{2}dx = \Big/\Big/\; \frac{d}{dx}\left(1+\frac{x^2}{a^2}\right)^{-2} = \left(1+\frac{x^2}{a^2}\right)^{-3}\cdot\frac{-4x}{a^2};$$

$$\frac{d}{dx}\left(\frac{-4x}{a^2}\left(1+\frac{x^2}{a^2}\right)^{-3}\right) = \frac{-4}{a^2}\left(1+\frac{x^2}{a^2}\right)^{-3} + \frac{24x^2}{a^4}\left(1+\frac{x^2}{a^2}\right)^{-4}\Big/\Big/ \Rightarrow$$

$$\Rightarrow -\frac{A^2\hbar^2}{2m}\int_{-\infty}^{\infty}\left(-\frac{4}{a^2}\left(1+\frac{x^2}{a^2}\right)^{-5} + \frac{24x^2}{a^4}\left(1+\frac{x^2}{a^2}\right)^{-6}\right)dx + \int_{-\infty}^{\infty}A^2\frac{m\omega^2}{2}x^2\left(1+\frac{x^2}{a^2}\right)^{-4}dx =$$

$$= \Big/\; -\frac{4}{a^2}dx \to -\frac{4x}{a^2}$$

$$\left(1+\frac{x^2}{a^2}\right)^{-5} \to -5\cdot\frac{2x}{a^2}\left(1+\frac{x^2}{a^2}\right)^{-6}\Big/\Big/$$

$$= -\frac{A^2\hbar^2}{2m}\int_{-\infty}^{\infty}\left(1+\frac{x^2}{a^2}\right)^{-6}\cdot\left(\frac{x^2(-16)}{a^4}\right) + \frac{A^2m\omega^2}{2}\int_{-\infty}^{\infty}\left(1+\frac{x^2}{a^2}\right)^{-4}x^2dx =$$

$= 0 = ...$ Полюс n-го порядку

Розклад в ряд Лорана!