$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$; $\hat{x} = x$;   Варіант 1.

№1

$[\hat{x}\hat{p}_x]\psi = \left[-i\hbar x\frac{\partial}{\partial x} + i\hbar\frac{\partial}{\partial x}\cdot x\right]\psi(x) = -i\hbar x\cdot\frac{\partial\psi}{\partial x} + i\hbar\frac{\partial}{\partial x}x\psi(x) =$

$= -i\hbar x\frac{\partial\psi}{\partial x} + i\hbar x\cdot\frac{\partial\psi}{\partial x} + i\hbar\cancel{\frac{\partial\psi}{\partial x}}\psi = i\hbar\psi.$

$\Rightarrow [\hat{x}\hat{p}_x] = i\hbar.$

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\varphi}\right)^2\psi(\rho,\varphi) = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\varphi}\right)\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial\psi}{\partial\varphi}\right) =$$

$$= \frac{1}{\rho}\cdot\left(-\frac{1}{\rho^2}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial\varphi\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\partial\varphi} + \frac{\partial^2\psi}{\partial\varphi^2} =$$

$$= -\frac{1}{\rho^3}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\varphi\partial\rho} + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\partial\varphi} + \frac{\partial^2\psi}{\partial\varphi^2} =$$

$$= \left[\frac{\partial^2}{\partial\varphi^2} + \frac{2}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi} - \frac{1}{\rho^3}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2}\right]\psi$$

Отже, $\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\varphi}\right)^2 = \left[\frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho^3}\frac{\partial}{\partial\rho} + \frac{2}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi} + \frac{\partial^2}{\partial\varphi^2}\right]$;

№2

$\psi(x) = C\exp\left(\frac{ip_x x}{\hbar}\right)\varphi(x)$;

$$\langle p_x\rangle = -i\hbar C^2\int_{-\infty}^{+\infty}\varphi\cdot\exp\left(\frac{ip_x x}{\hbar}\right)\frac{\partial}{\partial x}\exp\left(-\frac{ip_x x}{\hbar}\right)\varphi\,dx =$$

$$= +\frac{\hbar C^2 p_x}{\hbar}\int_{-\infty}^{+\infty}\varphi^2 dx - i\hbar C^2\int_{-\infty}^{+\infty}\varphi\cdot\frac{\partial\varphi}{\partial x}dx =$$

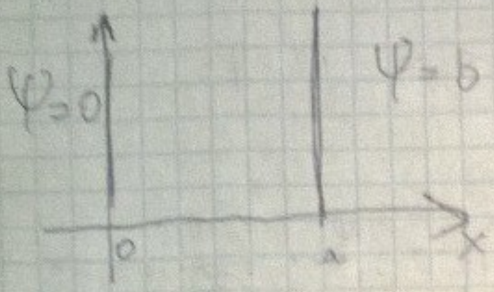
$$\int_{-\infty}^{+\infty}\varphi\frac{\partial\varphi}{\partial x}dx = \varphi\cdot\varphi\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{\partial\varphi}{\partial x}\varphi\,dx \Rightarrow \int_{-\infty}^{+\infty}\varphi\frac{\partial\varphi}{\partial x}dx = \frac{1}{2}\varphi\cdot\varphi\Big|_{-\infty}^{+\infty}$$

$$= \frac{\hbar C^2 p_x}{\hbar}\int_{-\infty}^{+\infty}\varphi^2 dx - \frac{i\hbar C^2}{2}\varphi^2\Big|_{-\infty}^{+\infty}; \quad - ?$$

аналогічно $\langle p_x^2 \rangle$.

№ 3

m, $\hat{V} = V_0\left(1 - \left|\frac{2x}{a} - 1\right|\right)$; a;



$\hat{H}\psi = E\psi$;

$V_0 = 0$, $0 \le x \le a$;

$V_0 \to \infty$, ~~x≠a~~ $\left[\begin{array}{l} x \ge a; \\ x \le 0; \end{array}\right.$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$;

$\frac{\partial^2\psi}{\partial x^2} + \underbrace{\frac{2mE}{\hbar^2}}_{=k^2}\psi = 0$; $\quad \lambda^2 + k^2 = 0$; $\lambda = \pm ik$

$\psi = C_1 e(-ikx) + C_2 e(ikx)$;

$\psi(0) = 0 \Rightarrow C_1 + C_2 = 0$; $\Rightarrow \psi_1 = C_2(-e^{-ikx} + e^{ikx}) =$

$= 2C_2 i \sin kx$

$\psi(a) = 0$; $\quad ka = \pi n$, $n \in Z$;

$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{\pi n}{a}$; $\quad E = \frac{\hbar^2\pi^2}{2mEa^2}n^2$;

$4C_2^2 \int_0^a \sin^2 k x = 1$; $\quad 2iC_2 \left. \left(-\frac{\cos kx}{k}\right)\right|_0^a = 1$;

$\frac{2iC_2}{k}\left(-(-1)^n + 1\right) =$

$\frac{4C_2^2}{2}\int_0^a 1 - \cos 2kx \, dx = C_2^2 2a = 1$;

$C_2 = \frac{1}{\sqrt{2a}}$;

$\psi_n = \frac{1}{\sqrt{2a}} \cdot 2 \sin kx = \sqrt{\frac{2}{a}} \sin k_n x$;

Збурення $\hat{V}$, , тоді $(\hat{H}_0 + \hat{V})\psi = E\psi$;

Стани невироджені

$$E_n' = E_n^0 + \lambda \tilde{V}_{nn} = E_n^{(1)} + \lambda^2 E_n^{(2)}; \quad E_n^{(2)} = \sum_{k \neq n} \frac{|\tilde{V}_{kn}|^2}{E_n - E_k}$$

$$\lambda^* \tilde{V} = V;$$

тоді $E_n^{(1)} = \int_0^a \psi_n V_0 \psi^* dx = \frac{2V_0}{a} \int_0^a \sin^2 k_n x \cdot \left(1 - \left|\frac{2x}{a} - 1\right|\right) =$

$$= \left\{ \frac{2x}{a} - 1 = 0; \quad 2x = a \Rightarrow x = \frac{a}{2}; \quad x \geq \frac{a}{2} > 0; \quad x \leq \frac{a}{2} < 0 \right\} \rightarrow$$

$\rightarrow$ маємо два звичайні інтеграли.

$E_n^{(0)} = \frac{\hbar^2 n^2 \pi^2}{2mEa^2}$ $\gg E^{(1)} \Rightarrow$ звідси критерій.

$N^0 = 4$

1S стан

$$\psi_{100} = \sqrt{\frac{Z^3}{\pi a^3}}\, e^{-\frac{Zr}{a}} \qquad Z = 1;$$

$$\psi_{100} = \sqrt{\frac{1}{\pi a^3}}\, e^{-\frac{r}{a}}; \quad a = \frac{\hbar^2}{\mu e^2};$$

ст. 110 – ізотопічний зсуви

ст. 215 – початок збурення:

$$W_{sm} = \frac{|V_{sm}(0)|^2}{\hbar^2 \omega_{sm}^2}; \quad \text{від } s \text{ до } m;$$

$$V_0 = -\frac{e^2}{r}; \qquad V' = -\frac{2e^2}{r};$$

$$\tilde{V} = V' - V_0 = -\frac{2e^2}{r} + \frac{e^2}{r} = -\frac{e^2}{r};$$

$$\omega_{sm} = \hbar^{-1}(E_s - E_m)$$

$$V_{sm}(0) = -\int \psi_{100} \frac{e^2}{r} \cdot 4\pi r^2 \cdot \psi_{200}^* dr \quad -?$$

$\Rightarrow \exp(0) = 1$

Варіант 2

① Довести рівність $[\vec{A}(\vec{r}), \hat{\vec{L}}] = i\hbar \vec{r} \cdot \text{rot} \vec{A}(\vec{r})$

$\hat{\vec{L}} = -i\hbar [\vec{r} \times \vec{\nabla}]$

$[\vec{A}(\vec{r}), \hat{\vec{L}}] = \vec{A}(\vec{r}) \hat{\vec{L}} - \hat{\vec{L}} \vec{A}(\vec{r})$

$$\vec{A}(\vec{r}) \hat{\vec{L}} = \vec{A}(\vec{r}) \cdot (-i\hbar) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} =$$

$$= -i\hbar \vec{A}(\vec{r}) (\vec{i} \cdot 0 - \vec{j} \cdot 0 - \vec{k} \cdot 0) = 0$$

$$\hat{\vec{L}} \cdot \vec{A}(\vec{r}) = -i\hbar [\vec{r} \times \vec{\nabla}] \cdot \vec{A}(\vec{r}) = -i\hbar \vec{A}(\vec{r}) \cdot [\vec{r} \times \vec{\nabla}] =$$

$$= -i\hbar \vec{r} \cdot [\vec{\nabla} \times \vec{A}(\vec{r})] = -i\hbar \vec{r} \cdot \text{rot} \vec{A}(\vec{r})$$

$$[\vec{A}(\vec{r}), \hat{\vec{L}}] = -\hat{\vec{L}} \vec{A}(\vec{r}) = i\hbar \vec{r} \cdot \text{rot} \vec{A}(\vec{r})$$

② Хвильова функція частинки має вигляд $\psi(x) = C \exp(i p_x x/\hbar) \varphi(x)$, де $\varphi(x)$ - дійсна функція. Знайти $\langle x \rangle$ та $\langle x^2 \rangle$.

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^* \cdot x \cdot \psi \, dx = \int_{-\infty}^{+\infty} C^2 \varphi^2(x) e^{-ip_x x/\hbar} x e^{ip_x x/\hbar} dx =$$

$$= \int_{-\infty}^{+\infty} x C^2 \varphi^2(x) dx = C^2 \int_{-\infty}^{+\infty} x \varphi^2(x) dx$$

$$\langle x^2 \rangle = C^2 \int_{-\infty}^{+\infty} x^2 \varphi^2(x) dx$$

④ Знайдіть наближене значення енергії основного стану гармонічного осцилятора варіаційним методом, використовуючи пробну функцію

$$\frac{\hbar^2}{2m}\left(\frac{\partial\psi_2}{\partial x}\Big|_0 - \frac{\partial\psi_1}{\partial x}\Big|_0\right) = -\alpha\,\psi(0);$$

$$\frac{\partial\psi_2}{\partial x}\Big|_0 - \frac{\partial\psi_1}{\partial x}\Big|_0 = \frac{-2\alpha m}{\hbar^2}\psi(0);$$

$$\frac{\partial\psi_2}{\partial x}\Big|_0 = -\cancel{2}\kappa C_2 e^{-\cancel{2}\kappa x}\Big|_0 = -\cancel{2}\kappa C_2;$$

$$\frac{\partial\psi_1}{\partial x}\Big|_0 = +\cancel{2}\kappa C_2 e^{\cancel{2}\kappa x}\Big|_0 = \cancel{2}\kappa C_2,\quad C_1 = C_2$$

$$\cancel{2}\kappa C_2 + \cancel{2}\kappa C_2 = 2\cancel{2}\kappa C_2 = \frac{2\alpha m}{\hbar^2}\psi(0).$$

$$C_2 = \frac{\cancel{2}\alpha m}{\cancel{2}\kappa\hbar^2}\psi(0) = \frac{\alpha m}{\cancel{2}\kappa\hbar^2}C_2;$$

$$\kappa = \frac{\alpha m}{\cancel{2}\hbar^2};$$

$$\int_{-\infty}^{+\infty}|\psi|^2\,dx = 1$$

$$\int_{-\infty}^{0} C_2^2 e^{+\cancel{2}\kappa x}\,dx + \int_{0}^{+\infty} C_2^2 e^{-\cancel{2}\kappa x}\,dx =$$

$$= \frac{C_2^2}{2\cancel{2}\kappa}\Big|_{-\infty}^{0} e^{2\kappa x} + \frac{C_2^2}{-2\kappa}\Big|_{0}^{+\infty} e^{-2\kappa x} = \frac{2C_2^2}{\kappa} = 1;$$

$$C_2 = \sqrt{\frac{\kappa}{\cancel{2}}} = \sqrt{\frac{\alpha m}{\cancel{2}\hbar^2}};$$

Отже, $\kappa = \frac{\alpha m}{\hbar^2}$; $C = \sqrt{\frac{\kappa}{\cancel{2}}} = \sqrt{\frac{\alpha m}{\cancel{2}\hbar^2}}$;

$$\psi_1 = \sqrt{\kappa}\,e^{+\kappa x};\quad \psi_2 = \sqrt{\kappa}\,e^{-\kappa x};$$

$$\langle U\rangle = \int_{-\infty}^{+\infty}\psi U\psi^*\,dx = \int_0^{+\infty}\psi_2 U\psi_2\,dx + \int_{-\infty}^{0}\psi_1 U\psi_1\,dx = -\alpha\cdot\psi^2(0) =$$

$$= -\alpha\cdot\kappa\cdot 1 = -\alpha\kappa = -\frac{\alpha^2 m}{\hbar^2};$$

Відповідь: $\langle U\rangle = -\frac{\alpha^2 m}{\hbar^2}$;

№ 2

$$[\hat p_x, [f(x,y,z), \hat p_x]];$$

$$[f(x,y,z),\hat p_x]\psi(x,y,z) = f\cdot\left(-i\hbar\frac{\partial}{\partial x}\right)\psi +$$

$$+ i\hbar\frac{\partial}{\partial x}f\psi = -if\hbar\frac{\partial\psi}{\partial x} + i\hbar\frac{\partial f}{\partial x}\psi +$$

$$+ i\hbar f\frac{\partial\psi}{\partial x} = i\hbar\frac{\partial f}{\partial x}\psi,\ \text{тоді}$$

$$[f(x,y,z),\hat p_x] = i\hbar\frac{\partial f}{\partial x};$$

тоді $[\hat p_x, i\hbar\frac{\partial f}{\partial x}]\psi = +\hbar^2\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\psi -$

Варіант 3

№1

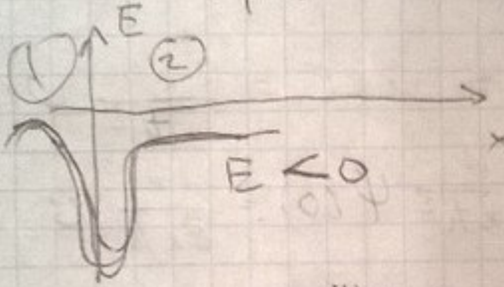
$U(x) = -\alpha\delta(x)$; $\langle U\rangle$ - ?

В ямі:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{2m|E|}{\hbar^2}\psi = 0; \quad k = \sqrt{\frac{2mE}{\hbar^2}};$$

$$\lambda^2 = -k^2; \quad \lambda = \pm k;$$

$$\psi = C_1 e^{+kx} + C_2 e^{-kx};$$



E<0

ф-я має бути скінченною, тоді

$$\psi_1 = C_1 e^{kx}; \quad |\psi| < \infty;$$

$$\psi_2 = C_2 e^{-kx};$$

$$\psi_1(0) = \psi_2(0) \Rightarrow C_1 = C_2.$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - (E-V)\psi = 0;$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + (V-E)\psi = 0;$$

$$+\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\partial^2\psi}{\partial x^2}dx = \int_{-\varepsilon}^{\varepsilon}(V-\overset{const}{E})\psi dx;$$

$\varepsilon \to 0$; ~~$\psi = const$~~

$$\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\psi}{\partial x} = -\int_{-\varepsilon}^{\varepsilon}\delta(x)\alpha\psi dx,$$

$$-\hbar^2 \frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x} - \hbar^2 \cdot \frac{\partial^2 f}{\partial x^2}\psi + \hbar^2 \frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x} -$$

$$-\hbar^2 \frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x} = \hbar^2 \frac{\partial^2 f}{\partial x^2}\psi;$$

Остаточно:

$$\int_0^{+\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}};$$

В: $[\hat{p}_x, [f(x,y,z), \hat{p}_x]] = \hbar^2 \frac{\partial^2 f}{\partial x^2};$

№ 4

$U(x) = F \cdot x;\quad x \geq 0;\quad U = \infty$ $\quad x \leq 0$, $\psi = 0$

$(\psi > 0)$

Пробна ф-я $\psi(x) = A \cdot x \cdot \exp(-\alpha x)$.

Умова нормування:

$$\int_0^{+\infty} A x e^{-\alpha x} \cdot A x e^{-\alpha x} dx = A^2 \int_0^{+\infty} x^2 e^{-2\alpha x} dx =$$

$$= \cancel{A^2 \int_{-\infty}^{0} x^2 e^{-2\alpha x} dx} \quad A^2 \cdot \frac{2}{(2\alpha)^3} = 1;$$

$$A^2 = 4\alpha^3; \quad A = \sqrt{4\alpha^3} = 2\sqrt{\alpha^3}$$

Знайдемо тоді $J(\alpha) = \int \psi^* \hat{H} \psi \, dx =$

$= \int \psi^* \hat{T} \psi \, dx + \int \psi^* \hat{U} \psi \, dx = \langle T \rangle +$

$+ \langle U \rangle$

№ 4

$$\langle T \rangle = \int \psi^* \hat{T} \psi \, dx = \int \psi^* \left(-\frac{\hbar^2}{2m}\Delta\right) \psi \, dx =$$

$$= \frac{1}{2\mu} \int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \cdot \underbrace{\left(-i\hbar \frac{\partial}{\partial x}\right) \psi}\, dx =$$

$$= \frac{1}{2\mu} \int -i\hbar \frac{\partial \psi}{\partial x} \cdot \psi \left(i\hbar \frac{\partial}{\partial x}\right) \cdot \psi^* \, dx \{$$ умова самоспряженості $$\} = \frac{1}{2\mu} \int \hat{p}_x \psi \cdot (\hat{p}_x \psi)^* \, dx =$$

$$= \frac{1}{2\mu} \int |\hat{p}_x \psi|^2 \, dx = \frac{1}{2\mu} \int \left| -i\hbar \frac{\partial \psi}{\partial x} \right|^2 dx =$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} A x e^{-\alpha x} = A e^{-\alpha x} - A \alpha x e^{-\alpha x} =$$

$$= A e^{-\alpha x} (1 - \alpha x)$$

$$= \frac{1}{2\mu} \int_0^{+\infty} \hbar^2 A^2 e^{-2\alpha x} (1 - \alpha x)^2 dx = A^2 \hbar^2 \cdot$$

$$\frac{1}{2\mu} \cdot \left(\left. \frac{e^{-2\alpha x}}{-2\alpha} \right|_0^{+\infty} - 2\alpha \cdot \frac{1}{(2\alpha)^2} + \frac{2\alpha^2}{(2\alpha)^3} \right) =$$

$$= \frac{A^2 \hbar^2}{2\mu} \left(\frac{1}{2\alpha} - \frac{1}{2\alpha} + \frac{1}{4\alpha} \right) = \frac{A^2 \hbar^2}{2\mu \cdot 4\alpha} =$$

$$= \frac{4\alpha^3 \cdot \hbar^2}{2 \cdot \mu \cdot 4\alpha} = \frac{\hbar^2 \alpha^2}{2\mu} = \frac{\hbar^2 \alpha^2}{2\mu}$$

5

$$\langle U\rangle = \int_0^{+\infty} A x e^{-\alpha x} F \cdot x \cdot A x \cdot e^{-\alpha x}\, dx =$$

$$= A^2F\int_0^{+\infty} x^3 e^{-2\alpha x}\, dx = A^2F\cdot\frac{6}{(2\alpha)^4} =$$

$$= 4\cdot\alpha^{3}\cdot F\cdot\frac{6}{16\cdot\alpha^4} = \frac{3}{2}F\cdot\alpha^{-1};$$

Отже,

$$\cancel{J(\alpha) = \frac{\hbar^2\sqrt{\alpha}}{2} + \frac{3}{4}F\alpha^{-2,5};}$$

$$\cancel{\frac{\partial J(\alpha)}{\partial\alpha} = 0; \quad \frac{\hbar^2}{2}\cdot\frac{1}{2\sqrt{\alpha}} + \frac{3}{4}F\cdot\frac{(-5)}{2}\cdot\alpha^{-3,5} =}$$

$$\cancel{= 0;}$$

$$\cancel{\frac{\hbar^2}{4}\frac{1}{\sqrt{\alpha}} - \frac{15F}{8}\frac{1}{\sqrt{\alpha^7}} = 0;}$$

$$J(\alpha) = \frac{\hbar^2\alpha^2}{2\mu} + \frac{3}{2}F\frac{1}{\alpha};$$

$$\frac{\partial J}{\partial\alpha} = 0; \quad \frac{\alpha\hbar^2}{\mu} - \frac{3}{2}F\cdot\frac{1}{\alpha^2} = 0;$$

$$\frac{\alpha^3\hbar^2}{\mu} = \frac{3}{2}F$$

$$\alpha^3\hbar^2 = \frac{3}{2}\mu F; \quad \alpha^3 = \frac{3\mu F}{2\hbar^2};$$

$$\alpha_0 = \sqrt[3]{\frac{3\mu F}{2\hbar^2}};$$

$$J(\alpha_0) = \frac{\hbar^2}{2\mu}\cdot\left(\frac{3\mu F}{2\hbar^2}\right)^{2/3} + \frac{3}{2}F\cdot\left(\frac{2\hbar^2}{3\mu F}\right)^{\frac{1}{3}} =$$

$$= (\hbar^2)^{\frac{1}{3}}\frac{3^{2/3}}{2\cdot 2^{2/3}}\cdot\mu^{-\frac{1}{3}}F^{2/3} + \frac{3}{2}F^{\frac{2}{3}}\left(\frac{3}{2}\right)^{-\frac{1}{3}}(\hbar^2)^{\frac{1}{3}}\cdot$$

$$\cdot\mu^{-\frac{1}{3}} = \left(\frac{\hbar^2}{\mu}\right)^{\frac{1}{3}}\left(\frac{3}{2}\right)^{\frac{2}{3}}\cdot F^{\frac{2}{3}}\left(\frac{1}{2}+1\right) =$$

$$= \frac{3}{2}\cdot\left(\frac{3}{2}\right)^{\frac{2}{3}}\cdot\left(\frac{\hbar^2F^2}{\mu}\right)^{\frac{1}{3}} = \left(\frac{243\,\hbar^2F^2}{3\ 2\mu}\right)^{\frac{1}{3}};$$

Відповідь: $E_0 = \left(\frac{243\,\hbar^2F^2}{32\mu}\right)^{\frac{1}{3}};$

(1) $$\int_V \Psi^*_{200}\, r\, \Psi\, dV = \int_0^r \underbrace{\frac{4\pi}{8\pi a^3}\left(1-\frac{r}{2a}\right)^2 e^{-\frac{r}{a}} r^3 dr}_{f(r)}$$

$$f(r) = \frac{1}{2a^3}\left(1-\frac{r}{a}+\frac{r^2}{4a^2}\right)e^{-\frac{r}{a}} r^3 dr =$$

$$= \frac{1}{2a^3}\left(r^3-\frac{r^4}{a}+\frac{r^5}{4a^2}\right)e^{-\frac{r}{a}}$$

$$\frac{\partial f(r)}{\partial r} = \frac{1}{2a^3}\left\{\left(3r^2-\frac{4r^3}{a}+\frac{5r^4}{4a^2}\right)e^{-\frac{r}{a}} - \right.$$

$$\left. -\frac{1}{a}e^{-\frac{r}{a}}\left(r^3-\frac{r^4}{a}+\frac{r^5}{4a^2}\right)\right\} = 0$$

$$3r^2-\frac{5r^3}{a}+\frac{9r^4}{4a^2}-\frac{r^5}{4a^3}=0$$

$$3-\frac{5r}{a}+\frac{9r^2}{4a^2}-\frac{r^3}{4a^3}=0$$

$$r^3-9ar^2+20a^2r-12a^3=0$$

(2) $[\hat{x}^3, \hat{T}]$ $\qquad \hat{T} = -\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$

$$\hat{x}^3 = x^3$$

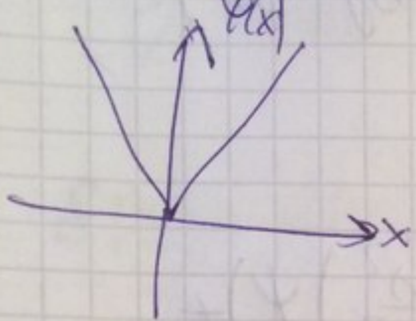
$$\hat{x}^3\hat{T}\psi - \hat{T}\hat{x}^3\psi = x^3\left(-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi\right) +$$

$$+\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)x^3\psi =$$

$$= -\cancel{\frac{\hbar^2}{2m}x^3\Delta\psi} + \frac{\hbar^2}{2m}\psi\frac{\partial^2(x^3)}{\partial x^2} + \cancel{\frac{\hbar^2}{2m}x^3\Delta\psi} =$$

$$= \frac{\hbar^2}{2m}\psi\frac{\partial}{\partial x}(3x^2) = \frac{\hbar^2}{2m}\psi 6x = \boxed{3\frac{\hbar^2}{m}x\psi}$$

(4) $\Psi(x) = C|x|$ $\quad \Psi(x) = A\exp\left(-\frac{x^2}{2\alpha}\right)$



Вар. метод.

1) $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + C|x|$

$\hat{H}\Psi = E\Psi$

Шукаємо розвязок у вигляді $\Psi(x) = A\exp\left(-\frac{x^2}{2\alpha}\right)$

2) Шукаємо інтеграл

$$y_0 = \int \Psi_0^*(x, \alpha_0, \beta_0)\hat{H}\Psi_0(x, \alpha_0, \beta_0)\,dx =$$

$$= \int_{-\infty}^{+\infty} A^2\exp\left(-\frac{x^2}{2\alpha}\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + C|x|\right)\exp\left(-\frac{x^2}{2\alpha}\right)dx =$$

$$= 2\int_0^{+\infty} A^2\exp\left(-\frac{x^2}{2\alpha}\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left[\exp\left(-\frac{x^2}{2\alpha}\right)\right] + C\,x\exp\left(-\frac{x^2}{2\alpha}\right)\right)dx =$$

$$= 2\int_0^{+\infty} A^2\exp\left(-\frac{x^2}{2a}\right)\left[-\frac{\hbar^2}{2m}\left\{\frac{x^2}{\alpha}e^{-\frac{x^2}{2\alpha}} - \frac{1}{\alpha}e^{-\frac{x^2}{2\alpha}}\right\} + C\,x\exp\left(-\frac{x^2}{2\alpha}\right)\right]dx =$$

$$= 2\int_0^{+\infty}\left[A^2\exp\left(-\frac{x^2}{2\alpha}\right)\left(-\frac{\hbar^2}{2m}\right)\cdot\frac{x^2}{\alpha}e^{-\frac{x^2}{2\alpha}} + A^2\exp\left(-\frac{x^2}{2\alpha}\right)\times \frac{1}{\alpha}\exp\left(-\frac{x^2}{2\alpha}\right)\cdot\frac{\hbar}{2m} + A^2 C\,x\exp^2\left(-\frac{x^2}{2\alpha}\right)\right]dx =$$

$$= 2\int_0^{+\infty} A^2 \frac{x^2}{\alpha^2}\left(-\frac{\hbar^2}{2m}\right) e^{-\frac{x^2}{\alpha}} dx + 2\int_0^{+\infty} A^2 \frac{\hbar^2}{2m}\frac{1}{\alpha} e^{-\frac{x^2}{\alpha}} dx +$$

$$+ 2\int_0^{\infty} A^2 C x\, e^{-\frac{x^2}{\alpha}} dx$$

Табличні інтеграли

$$\int_{-\infty}^{+\infty} A^2 e^{-\frac{x^2}{\beta^2}} dx \Rightarrow \qquad \alpha = \beta^2 \qquad \beta = \sqrt{\alpha}$$

$$\Rightarrow \int_0^{\infty} e^{-\frac{x^2}{\beta^2}} dx = \frac{1}{2A} \Rightarrow A^2\sqrt{\pi\beta^2} = 1 \Rightarrow A$$

$$y_0 = \frac{2A^2}{\beta^2}\left(-\frac{\hbar^2}{2m}\right)\cdot\frac{1}{4}\sqrt{\pi}\,\beta^{3/2} + 2A^2\frac{\hbar^2}{2m}\frac{1}{\beta^2}\sqrt{\pi\beta^2}$$

В общем может быть ошибка в интеграле

Дальше подставляем наше A, находим $y_0 = y(\alpha)$

Тогда $\frac{\partial y_0}{\partial \alpha} \Rightarrow \alpha_{опт} \Rightarrow \Psi_0 = \Psi_0(x, \alpha_{опт})$

№ 52

$$x \cdot \frac{d^2}{dx^2} \cdot x = \hat{\varphi};$$

Умова самоспряженості $\int \psi_n^*(x)\, \hat{\varphi}\, \psi_k(x)\, dx = \int \psi_k(x)\, \hat{\varphi}^* \psi_n^*(x)\, dx;$

$$\int \psi_n^* \cdot x \cdot \frac{d^2}{dx^2}(x \cdot \psi_k(x))\, dx = \underbrace{\frac{d(x\psi_k)}{dx} \cdot \psi_n^* x \Big|_{-\infty}^{+\infty}}_{=0} -$$

$$- \int_{-\infty}^{+\infty} \frac{d}{dx}(\psi_n^* \cdot x) \cdot \frac{d}{dx}(x \psi_k(x))\, dx =$$

$$= - \int_{-\infty}^{+\infty} \frac{d}{dx}(\psi_n^* \cdot x) \cdot \frac{d}{dx}(x \psi_k)\, dx = - \underbrace{x\psi_k \cdot \frac{d}{dx}(\psi_n^* \cdot x) \Big|_{-\infty}^{+\infty}}_{=0} +$$

$$+ \int_{-\infty}^{+\infty} x \cdot \psi_k \cdot \frac{d^2}{dx^2}(\psi_n^* \cdot x)\, dx = \int_{-\infty}^{+\infty} \psi_k\, x \cdot \frac{d^2}{dx^2}\, x\, \psi_n^*\, dx$$

тобто,

$$\hat{\varphi}^+ = x \frac{d^2}{dx^2} \cdot x$$

Відповідь: $\hat{\varphi}^+ = x \frac{d^2}{dx^2} x$;

(Гречко 11, 13)

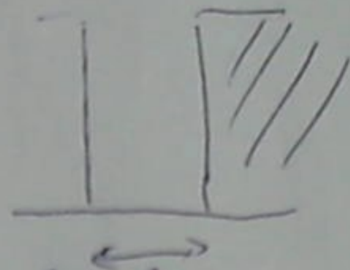
$$\int_{-\infty}^{+\infty} |\psi_{1,2}|^2 dx = 0 \Rightarrow \psi_{1,2}(\pm\infty) = 0.$$

$t = 0$

$\psi(x) = C \sin(3\pi x / a)$

$\psi(x,t) - ?$

$T - ?$



$$\int_0^a \psi(x)\psi^*(x)\,dx = C^2 \int_0^a \sin^2 \frac{3\pi x}{a}\,dx = C^2 \int_0^a \frac{1-\cos\frac{6\pi x}{a}}{2}\,dx =$$

$$= C^2 \frac{a}{2} = 1 \Rightarrow C = \sqrt{\frac{2}{a}}$$

Запишемо нестаціонарне Рівняння Шредінгера

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}(x,t)\,\psi(x,t)$$

Вважаємо $\hat{H} = -\frac{\hbar^2}{2m}\Delta + Const \to 0$

В одновимірному випадку

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$

Методом Розділення змінних $\psi(x,t) = X(x)\,T(t)$

$$i\hbar T' X + \frac{\hbar^2}{2m} T X'' = 0 \quad | : \hbar X T$$

$$i\frac{T'}{T} = -\frac{\hbar}{2m}\frac{X''}{X} = \lambda^2$$

$$T' + i\lambda^2 T = 0 \Rightarrow (\ln T)' = -i\lambda^2 \Rightarrow T = C_n e^{-i\lambda^2 t}$$

$i\hbar \frac{\partial}{\partial t} \quad H(x,t)\,\Psi(x,t)$

Вважаємо $\hat{H} = -\frac{\hbar^2}{2m}\Delta + Const_{\to 0}$

В одновимірному випадку

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi$$

Методом розділення змінних $\Psi(x,t) = X(x)T(t)$,

$$i\hbar T'X + \frac{\hbar^2}{2m}TX'' = 0 \quad \Big|: \hbar X T$$

$$i\frac{T'}{T} = -\frac{\hbar}{2m}\frac{X''}{X} = \lambda^2$$

$$\begin{cases} T' + i\lambda^2 T = 0 \\ \frac{\hbar^2}{2m}X'' - \lambda^2 X = 0 \end{cases} \Rightarrow \begin{matrix} (\ln T)' = -i\lambda^2 \\ \ln T = -i\lambda^2 t + C \end{matrix} \Rightarrow T = C_2 e^{-i\lambda^2 t}$$

$$X'' + \frac{2m\lambda^2}{\hbar}X = 0$$

$$X = A\sin\sqrt{\frac{2m}{\hbar}}\lambda x + B\cos\sqrt{\frac{2m}{\hbar}}\lambda x$$

При $t = 0$ $\Psi(x,0) = XT(0) = C_2 A\sin\sqrt{\frac{2m}{\hbar}}\lambda x + \underbrace{C_2 B\cos\sqrt{\frac{2m}{\hbar}}\lambda x}_{=0 \Rightarrow B = 0}$

$$\Rightarrow C_2 A = \sqrt{\frac{2}{a}}, \quad \sqrt{\frac{2m}{\hbar}}\lambda = \frac{3\pi}{a}$$

$$\Rightarrow \lambda = \sqrt{\frac{\hbar}{2m}}\frac{a}{3\pi}$$

① $\psi_{210}(r,\theta) = \frac{r\cos\theta}{4\sqrt{2\pi a^5}} e^{-\frac{r}{2a}}$

$\cancel{\frac{d\psi}{dr} = \frac{\cos\theta}{4\sqrt{2\pi a^5}} e^{-\frac{r}{2a}}\left(1 - \frac{r}{2a}\right) = 0 \Rightarrow \underline{r = 2a}}$

$\cancel{E_k = J \cdot \omega^2}$ $\qquad J = 2am$

$\cancel{\omega = \frac{v}{r} = \frac{p}{mr} =}$

$$\hat{p} = -i\hbar\nabla = -i\hbar\left(\bar{e}_r\frac{\partial}{\partial r} + \bar{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \bar{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$\langle T\rangle = \int_V \psi_{210}^* \hat{T}\psi_{210}\, dV = \frac{-\hbar^2}{2m16\cdot 2\pi a^5}\int_0^{2\pi} d\varphi\int_0^{\pi}\sin\theta\, d\theta\int_0^{\infty} r^2 dr\Big\{$$

$$\Big\{ r\cos\theta e^{-\frac{r}{2a}} \cdot \Big\{\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(r\cos\theta e^{-\frac{r}{2a}}\right)\right)}_{= f_1} + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta \cdot \frac{\partial}{\partial\theta}\left(r\cos\theta e^{-\frac{r}{2a}}\right)\Big)}_{= f_2}\Big\}\Big\} =$$

$$= \frac{-\hbar^2 2\pi}{64 m\pi a^5}\int_0^{\pi}\sin\theta\cos^2\theta\, d\theta\int_0^{\infty} e^{-\frac{r}{a}}\left(2r - \frac{r^2}{2a} - \frac{3r^2}{2a} + \frac{r^3}{4a^2} - 2r\right) dr$$

$$\int_0^{\pi}\cos^2\theta\, d\cos\theta = \frac{\cos^3\theta}{3}\Big|_0^{\pi} = \frac{-1-1}{3} = -\frac{2}{3} \qquad = 1$$

$$f_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(r\cos\theta\, e^{-\frac{r}{2a}}\right)\right) = \frac{\cos\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\left(e^{-\frac{r}{2a}} - \frac{r}{2a}e^{-\frac{r}{2a}}\right)\right) =$$

$$= \frac{\cos\theta}{r^2}\left(2r e^{-\frac{r}{2a}} - \frac{r^2}{2a}e^{-\frac{r}{2a}} - \frac{3r^2}{2a}e^{-\frac{r}{2a}} + \frac{r^3}{4a^2}e^{-\frac{r}{2a}}\right) =$$

$$= \cos\theta\, e^{-\frac{r}{2a}}\left(\frac{2}{r} - \frac{1}{2a} - \frac{3}{2a} + \frac{r}{4a^2}\right)$$

$$f_2 = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(r\cos\theta\, e^{-\frac{r}{2a}}\right)\right) =$$

$$= \frac{1\, r\, e^{-\frac{r}{2a}}}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right) = -\frac{e^{-\frac{r}{2a}}}{r}\frac{1}{\sin\theta}2\sin\theta\cos\theta =$$

$$= -\frac{e^{-\frac{r}{2a}}}{r}2\cos\theta$$

$$J = \int_0^\infty e^{-\frac{r}{a}}\left(-\frac{2r^2}{a} + \frac{r^3}{4a^2}\right)dr = -\frac{2}{a}\cdot\frac{2!}{\left(\frac{1}{a}\right)^3} + \frac{1}{4a^2}\cdot\frac{3!}{\left(\frac{1}{a}\right)^4} =$$

$$= -4a^2 + \frac{3}{2}a^2 = -\frac{5}{2}a^2$$

$$\langle T\rangle = \frac{-\hbar^2}{32\, m a^5}\cdot\frac{2}{3}\cdot\frac{5}{2}\cdot a^2 = -\frac{5\hbar^2}{96 a^3}$$

№: 5.2

$$x \cdot \frac{d^2}{dx^2} \cdot x = \hat{\varphi};$$

Умова самоспряженості $\int \psi_n^*(x)\, \hat{\varphi}\, \psi_k(x)\, dx = \int \psi_k(x)\, \hat{\varphi}^* \psi_n^*(x)\, dx;$

$$\int \psi_n^* \cdot x \cdot \frac{d^2}{dx^2}\left(x \cdot \psi_k(x)\right) dx = \underbrace{\frac{d(x\psi_k)}{dx} \cdot \psi_n^* x \Big|_{-\infty}^{+\infty}}_{=0} -$$

$$- \int_{-\infty}^{+\infty} \frac{d}{dx}\left(\psi_n^* \cdot x\right) \cdot \frac{d}{dx}\left(x\psi_k(x)\right) dx =$$

$$= - \int_{-\infty}^{+\infty} \frac{d}{dx}\left(\psi_n^* \cdot x\right) \cdot \frac{d}{dx}\left(x\psi_k\right) dx = - \underbrace{\left. x\psi_k \cdot \frac{d}{dx}\left(\psi_n^* \cdot x\right) \right|_{-\infty}^{+\infty}}_{=0} +$$

$$+ \int_{-\infty}^{+\infty} x \cdot \psi_k \cdot \frac{d^2}{dx^2}\left(\psi_n^* \cdot x\right) dx = \int_{-\infty}^{+\infty} \psi_k\, x \cdot \frac{d^2}{dx^2}\, x\, \psi_n^*\, dx$$

тобто,

$$\hat{\varphi}^+ = x \frac{d^2}{dx^2} \cdot x$$

Відповідь: $\hat{\varphi}^+ = x \frac{d^2}{dx^2} x;$

(Гречко 11, 13)

$$\int_{-\infty}^{+\infty} |\psi_{1,2}|^2 dx = 0 \Rightarrow \psi_{1,2}(\pm\infty) = 0.$$

1 $\hat{f} = \frac{d^2}{dx^2} - 4x^2$, $\psi(x) = \exp(-x^2)$

$$\hat{f}\psi = \left(\frac{d^2}{dx^2} - 4x^2\right) e^{-x^2} = \frac{d}{dx}\left(-2x e^{-x^2}\right) - 4x^2 e^{-x^2} =$$

$$= -2e^{-x^2} + \cancel{4x^2 e^{-x^2}} - \cancel{4x^2 e^{-x^2}} = -2e^{-x^2} = f\psi$$

де $f = -2 \Rightarrow \psi$ - власна функція оператора $\hat{f}$, а $f = -2$ його власне значення

② $\hat{A} = \left(x + \frac{d}{dx}\right)^2$, $\hat{B} = x\frac{d^2}{dx^2}$

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} =$$

$$= \left(x + \frac{d}{dx}\right)^2 \left(x\frac{d^2}{dx^2}\right) - x\frac{d^2}{dx^2}\left(x + \frac{d}{dx}\right)^2$$

$$\hat{A}\hat{B} = \left(x^2 + x\frac{d}{dx} + 1 + \frac{d^2}{dx^2}\right)\left(x\frac{d^2}{dx^2}\right) =$$

$$= x^3\frac{d^2}{dx^2} + x\left(\frac{d^2}{dx^2} + x\frac{d^3}{dx^3}\right) + x\frac{d^2}{dx^2} +$$

$$+ \frac{d}{dx}\left(\frac{d^2}{dx^2} + x\frac{d^3}{dx^3}\right) =$$

$$= x^3\frac{d^2}{dx^2} + x\frac{d^2}{dx^2} + x^2\frac{d^3}{dx^3} + x\frac{d^2}{dx^2} + \frac{d^3}{dx^3} +$$

$$+ \frac{d^3}{dx^3} + x\frac{d^4}{dx^4} = (x^3 + 2x)\frac{d^2}{dx^2} + (x^2+2)\frac{d^3}{dx^3} +$$

$$+ x\frac{d^4}{dx^4}$$

$$\hat{B}\hat{A} = x\frac{d^2}{dx^2}\left(x^2 + x\frac{d}{dx} + 1 + \frac{d^2}{dx^2}\right) =$$

$$= 2x + x\left(\frac{d}{dx}\left(\frac{d}{dx} + x\frac{d^2}{dx^2}\right)\right) + x\frac{d^4}{dx^4} =$$

$$= 2x + x\frac{d^2}{dx^2} + x^2\frac{d^3}{dx^3} + x\frac{d^2}{dx^2} + x\frac{d^4}{dx^4} =$$

$$= 2x + 2x\frac{d^2}{dx^2} + x^2\frac{d^3}{dx^3} + x\frac{d^4}{dx^4}$$

$[\hat{A},\hat{B}] = x^3\frac{d^2}{dx^2} + 2\frac{d^3}{dx^3} - 2x \Rightarrow$ не комутують

$$= -2e^{-x^2} + 4x^2e^{-x^2} - 4x^2e^{-x^2} = -2e^{-x^2} = f\psi$$

де $f = -2 \Rightarrow \psi$ - власна функція оператора $\hat{f}$, а $f = -2$ його власне значення

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③ Дано:

$U(x) = U_0[1 - |2x/a - 1|]$

m, a

ΔE_n - ? критерій - ? (у першому порядку)

Для такої ями без збурення

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

У першому наближенні $E_n^{(1)}$ має наступний вигляд:

$$E_n^{(1)} = \int_0^a \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} U_0[1 - |2x/a - 1|] \sqrt{\frac{2}{a}} \cdot \sin\frac{n\pi x}{a} dx = \frac{2}{a} U_0 \int_0^a [1 - |2x/a - 1|] \sin^2\frac{n\pi x}{a} dx$$

$$= \frac{2u_0}{a}\left(2\int_{a/2}^{a} \sin^2\frac{n\pi x}{a}dx + \int_0^{a/2}\frac{2x}{a}\sin^2\frac{n\pi x}{a}dx - \right.$$

$$\left. - \int_{a/2}^{a}\frac{2x}{a}\sin^2\frac{n\pi x}{a}dx\right) =$$

$$\int \sin^2\frac{n\pi x}{a}dx = \int\frac{1}{2}\left(1-\cos\left(\frac{2n\pi x}{a}\right)\right)dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{a}{2n\pi}\sin\frac{2n\pi x}{a}$$

$$\int x\sin^2\frac{n\pi x}{a}dx = \left| \begin{array}{l} u = x \\ du = dx \\ dv = \sin^2\frac{n\pi x}{a}dx \\ v = \frac{1}{2}x - \frac{1}{2}\frac{a}{2n\pi}\cdot\sin\frac{2\pi n x}{a} \end{array}\right| =$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\frac{ax}{2n\pi}\sin\frac{2n\pi x}{a} - \cancel{\ldots}\ \frac{1}{4}x^2 + \frac{1}{2}\frac{a}{2n\pi}\cdot\frac{a}{2n\pi}\cdot$$

$$\cdot\left(-\cos\frac{2n\pi x}{a}\right) = \frac{1}{4}x^2 - \frac{ax}{4n\pi}\sin\frac{2n\pi x}{a} - \frac{a^2}{8n^2\pi^2}\cdot$$

$$\cdot\cos\frac{2n\pi x}{a}$$

$$= \frac{2u_0}{a}\left(2\left(\frac{a}{4} - \frac{1}{2}\frac{a}{2n\pi}(0-0)\right) + \right.$$

$$= a/2$$

$$+ \frac{2}{a}\left(\frac{1}{4}\cdot\frac{a^2}{4} - \frac{a^2}{8n^2\pi^2}\cdot\cos n\pi + \frac{a^2}{8n^2\pi^2}\right) -$$

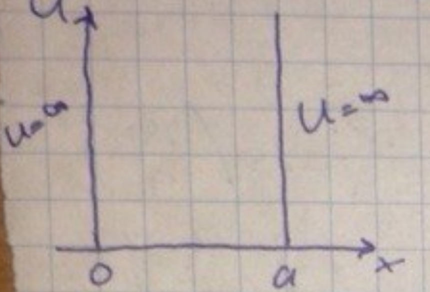
$$\left. - \frac{2}{a}\left(\frac{1}{4}\cdot\frac{3a^2}{4} - \frac{a^2}{8n^2\pi^2}(\cos 2n\pi - \cos n\pi)\right)\right) =$$

$$= \frac{2u_0}{a}\left(\frac{a}{2} + \frac{a}{8} - \frac{a}{4n^2\pi^2}(-1)^n + \frac{a}{4n^2\pi^2} - \frac{3a}{8} + \right.$$

$$\left. + \frac{a}{4n^2\pi^2}\left(1-(-1)^n\right)\right) = \frac{2u_0}{a}\left(\frac{a}{4} - \frac{a}{4n^2\pi^2}\left(1-2(-1)^n\right)\right)$$

15.1

(1) Частинка знаходиться в одновимірній нескінченно глибокій ~~яма~~ потенціальній ямі шириною a в основному квантовому стані.
Знайти ймовірність її знаходження в області
$a/4 \le x \le x/2$



Рівняння Шредінгера матиме вигляд:

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}(E-U)\psi=0$$

Ймовірність знаходження частинки за межами ями дорівнює 0, тому з умови неперервності випливає, що хвильова ф-я на межах ями дорівнює 0:

$$\begin{cases}\psi(0)=0\\ \psi(a)=0\end{cases}$$

В області $[0,a]$ р-ня Шредінгера прийме вигляд:

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}E\psi=0$$

Позначимо $\frac{2mE}{\hbar^2}=\omega^2$ $\qquad \psi''+\omega^2\psi=0$

Розвязок цього р-ня: $\psi(x)=A\sin(\omega x+\alpha)$

Розглянемо межові умови:

$$\begin{cases}\psi(0)=0\\ \psi(a)=0\end{cases} \qquad \begin{cases}A\cdot\sin\alpha=0\\ A\cdot\sin(\omega a+\alpha)=0\end{cases} \Rightarrow \underline{\alpha=0}$$

$$A\cdot\sin(\omega a)=0 \Rightarrow \omega a=\pm n\pi \quad (n=1,2\ldots)$$

$$\omega=\pm\frac{\pi n}{a} \qquad \psi_n(x)=A\sin\left(\frac{\pi n x}{a}\right)$$

Використаємо умову нормування:

$$\int_0^a|\psi(x)|^2dx=1 \Rightarrow A^2\int_0^a\sin^2\left(\frac{\pi n x}{a}\right)dx=1$$

$$\int_0^a \sin^2(kx)\,dx = \int_0^a \frac{1-\cos(2kx)}{2}\,dx = \int_0^a \frac{dx}{2} - \frac{1}{2}\int_0^a \cos(2kx)\,dx =$$

$$= \frac{x}{2}\Big|_0^a - \frac{1}{2}\cdot\frac{1}{2k}\sin(2kx)\Big|_0^a = \frac{a}{2} - \frac{1}{4k}\sin(2ka)$$

$$k = \frac{\pi n}{a} \Rightarrow \int_0^a \sin^2\left(\frac{\pi n}{a}x\right)dx = \frac{a}{2}$$

$$A^2\frac{a}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{a}} \Rightarrow \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

Ймовірність знаходження на проміжку $\left[\frac{a}{4}, \frac{a}{2}\right]$:

$$P_n = \int_{a/4}^{a/2} |\psi_n(x)|^2 dx = \int_{a/4}^{a/2} \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)dx =$$

$$= \frac{2}{a}\int_{a/4}^{a/2} \sin^2\left(\frac{n\pi x}{a}\right)dx = \cancel{\frac{2}{a}\left(\frac{a}{2} - \frac{1}{4k}\sin 2\right)}$$

$$= \cancel{\frac{2}{a}\left(\frac{a}{2} - \frac{1\cdot a}{4n\pi}\sin\right)} \frac{2}{a}\left(\frac{x}{2} - \frac{1}{4k}\sin(2kx)\right)\Big|_{a/4}^{a/2} =$$

$$= \frac{2}{a}\left(\frac{x}{2} - \frac{1\cdot a}{4\pi n}\sin\left(\frac{2\pi n}{a}x\right)\right)\Big|_{a/4}^{a/2} = \frac{2}{a}\left(\frac{a}{4} - \frac{a}{8} -\right.$$

$$\left. - \frac{a}{4\pi n}\underbrace{\sin(\pi n)}_{=0} + \frac{a}{4\pi n}\sin\left(\frac{\pi n}{2}\right)\right) =$$

$$= \frac{2}{a}\left(\frac{a}{8} + \frac{a}{4\pi n}\cdot\sin\left(\frac{\pi n}{2}\right)\right) = \frac{1}{4} + \frac{1}{2\pi n}\cdot\sin\left(\frac{\pi n}{2}\right)$$

Якщо $n = 2k$ (парне), то $P_n = \frac{1}{4}$

Якщо $n = 2k+1$ (непарне), то $P_n = \frac{1}{4} + \frac{1}{2\pi n}(-1)^k$

15.2

2) Показати, що для довільного оператора $\hat{A}$ маємо: $\langle \hat{A}^2 \rangle \geq 0$

Оператор, за допомогою якого визначається середнє значення фізичної величини λ, повинен бути самоспряженим, тобто $\hat{A} = \hat{A}^+$.
Розглянемо квадрат $\hat{A}$, тобто $\langle \hat{A}^2 \rangle$:

$$\langle \hat{A}^2 \rangle = \int \psi^* \hat{A}^2 \psi \, d\tau$$

Використаємо умову самоспряженості:

$$\int \psi_1^* \hat{A} \psi_2 \, d\tau = \int \psi_2 (\hat{A}\psi_1)^* d\tau$$

Позначимо $\hat{A}\psi_1 = \psi_2$, тоді:

$$\langle \hat{A}^2 \rangle = \int \psi_1^* \hat{A} \psi_2 \, d\tau = \int \psi_2 \underbrace{(\hat{A}\psi_1)}_{\psi_2}{}^* d\tau = \int \psi_2 \psi_2^* \, d\tau \geq 0$$

так як $|\psi_2|^2 \geq 0$