Домашня робота №8

N454

$$u_t = u_{xx} + \theta(t)\delta(x)$$

$$u(x,0) = \delta(x-x_0)$$

$u(x,t) = \mathcal{E}(x,t) * F(x,t) = \varepsilon(x,t) * \theta(t)\delta(x) +$
$+ \mathcal{E}(x,t) * \delta(x-x_0)\delta(t)$, де $\varepsilon(x,t) =$
$= \frac{\theta(t)}{2\sqrt{\pi t}} \exp(-x^2/4t)$, при $n=1$, $a=1$

Розрахуємо першу згортку:

$(\varepsilon(x,t) * \theta(t)\delta(x), \varphi(x,t)) = (\theta(t)\delta(x), (\varepsilon$
$(x',t'), \varphi(x+x', t+t')))$;

Зробимо заміну $x+x' = z$, $t+t' = \tau$

$(\theta(t)\delta(x), (\varepsilon(x',t'), \varphi(x+x', t+t'))) =$
$= (\varphi(z,\tau), (\theta(t)\delta(x), \varepsilon(z-x, \tau-t)))$;

Після інтегрування за змінною x:

$(\varphi(z,\tau), (\theta(t)\delta(x), \varepsilon(z-x, \tau-t))) =$
$= (\varphi(z,\tau), (\theta(t), \varepsilon(z, \tau-t))) =$
$= \left(\varphi(z,\tau), \left(\theta(t), \frac{\theta(\tau-t)}{2\sqrt{\pi(\tau-t)}} \exp\left(-\frac{z^2}{4(\tau-t)}\right)\right)\right)$;

Виконуючи інтегрування за змінною t в інтервалі від 0 до τ, маємо:

$$I(\tau, z) = \frac{1}{2\sqrt{\pi}} \int_0^{\tau} \frac{\exp[-z^2/4(\tau - t)]}{\sqrt{\tau - t}} dt =$$

$$= \left| y = \frac{|z|}{2\sqrt{\tau - t}},\ dt = \frac{z^2 dy}{2y^3} \right| = \frac{|z|}{2\sqrt{\pi}} \int_{\frac{|z|}{2\sqrt{\tau}}}^{\infty} \frac{\exp(-y^2)}{y^2} dy$$

$$= \left| \begin{matrix} u = \exp(-y^2), & du = -2y\exp(-y^2)dy \\ dv = \frac{dy}{y^2} & v = \frac{-1}{y} \end{matrix} \right| =$$

$$= \frac{|z|}{2\sqrt{\pi}} \left(-\frac{\exp(-y^2)}{y} \Big|_{\frac{|z|}{2\sqrt{\tau}}}^{\infty} - 2 \int_{\frac{z}{2\sqrt{\tau}}}^{\infty} \exp(-y^2) dy \right) =$$

$$= \sqrt{\frac{\tau}{\pi}} \exp\left(-\frac{z^2}{4\tau}\right) - \frac{|z|}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{\tau}}}^{\infty} \exp(-y^2) dy =$$

$$= \sqrt{\frac{\tau}{\pi}} \exp\left(-\frac{z^2}{4\tau}\right) - \frac{|z|}{2} + \frac{|z|}{2} \operatorname{erf}\left(\frac{|z|}{2\sqrt{\tau}}\right),$$

де $\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \Rightarrow$

$\Rightarrow \mathcal{E}(x,t) * \theta(t)\delta(x) = \theta(t)\left[\sqrt{\frac{t}{\pi}} \exp\left(-\frac{x^2}{4t}\right) - \frac{|x|}{2} \operatorname{erfc} \left(\frac{|z|}{2\sqrt{t}}\right)\right]$, де $\operatorname{erfc} x = 1 - \operatorname{erf} x$

Розрахуємо другу згортку:

$$(\mathcal{E}(x,t) * \delta(x-x_0)\,\delta(t), \varphi(x,t)) =$$
$$= (\delta(x-x_0)\,\delta(t), (\mathcal{E}(x',t'), \varphi(x+x',t+t'))) =$$
$$= (\varphi(z,\tau), (\delta(x-x_0)\,\delta(t), \mathcal{E}(z-x,\tau-t))) =$$
$$= (\varphi(z,\tau), \mathcal{E}(z-x_0,\tau)) = (\mathcal{E}(x-x_0,t), \varphi(x,t));$$

$$\mathcal{E}(x,t) * \delta(x-x_0)\,\delta(t) = \frac{\theta(t)}{2\sqrt{\pi t}} \exp\left[\frac{-(x-x_0)^2}{4t}\right].$$

$$u(x,t) = \theta(t)\left\{\sqrt{\frac{t}{\pi}} \exp\left(-\frac{x^2}{4t}\right) - \frac{|x|}{2}\operatorname{erfc}\left(\frac{|x|}{2\sqrt{t}}\right) + \frac{1}{2\sqrt{\pi t}}\exp\left[\frac{-(x-x_0)^2}{4t}\right]\right\}$$

Перевірка:

$$\lim_{t\to 0} \frac{\theta(t)}{2\sqrt{\pi t}} \exp\left[\frac{-(x-x_0)^2}{4t}\right]$$

дорівнює нескінченності, якщо $x = x_0$

нулю, якщо $x \neq x_0$

$$u_t(x,t) = \delta(t)\,\delta(x-x_0) + \theta(t)\left\{\frac{1}{2\sqrt{\pi t}} \cdot \exp\left(-\frac{x^2}{4t}\right) - \frac{1}{4t\sqrt{\pi t}}\left[1 - \frac{(x-x_0)^2}{2t}\right]\exp\left[\frac{-(x-x_0)^2}{4t}\right]\right\};$$

$$u_x(x,t) = \theta(t)\left\{-\frac{\operatorname{sign} x}{2}\operatorname{erfc}\left(\frac{|x|}{2\sqrt{t}}\right) -\right.$$

$$-\frac{(x-x_0)}{4t\sqrt{\pi t}}\exp\left[-\frac{(x-x_0)^2}{4t}\right]\Big\}.$$

$$u_{xx}(x,t) = -\theta(t)\delta(x) + \theta(t)\Big\{\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right) -$$

$$-\frac{1}{4t\sqrt{\pi t}}\left[1-\frac{(x-x_0)^2}{2t}\right]\exp\left[\frac{-(x-x_0)^2}{4t}\right]\Big\}.$$

Виходячи з цього: $u_t(x,t) = u_{xx}(x,t) +$

$+ \theta(t)\delta(x) + \delta(t)\delta(x-x_0).$

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$$\begin{cases} u_t = u_{xx} + e^{\alpha t}\delta(x) \\ u(x,0) = \delta(1-|x|) \end{cases}$$

$$u(x,t) = \varepsilon_1(x,t) * \delta(x)\theta(t)\exp(\alpha t) +$$

$$+ \varepsilon_1(x,t) * \delta(1-|x|)\delta'(t)$$

$$\varepsilon_1(x,t) = \frac{\theta(t-|x|)}{2}.$$

$$(\varepsilon_1(x,t) * \delta(x)\theta(t)\exp(\alpha t), \varphi(x,t)) =$$

$$= (\delta(x)\theta(t)\exp(\alpha t), (\varepsilon_1(x',t'), \varphi(x+x', t+t'))) =$$

$$= (\varphi(z,\tau), (\theta(t)\exp(\alpha t), \varepsilon_1(z, \tau - t))) =$$

$$= \left(\frac{\varphi(z,\tau)}{2}, (\theta(t)\exp(\alpha t), \theta(\tau - t - |z|))\right) =$$

$= \left(\frac{1}{2\alpha} \left\{ \exp[\alpha (t - |x|)] - 1 \right\} \theta(t - |x|), \varphi(x,t) \right)$

Далі наступна згортка:

$(\mathcal{E}_1(x,t) * \delta(1-|x|)\delta''(t), \varphi(x,t)) = (\delta(1-|x|)\delta''(t), (\mathcal{E}_1(x',t'), \varphi(x+x', t+t'))) = (\varphi(z,\tau), ([\delta(x-1) + \delta(x+1)]\delta''(t), \mathcal{E}_1(z-x, \tau-t))) = (\varphi(z,\tau), (\delta''(t), \mathcal{E}_1(z-1, \tau-t))) + (\varphi(z,\tau), (\delta''(t), \mathcal{E}_1(z+1, \tau-t))) = -(\varphi(z,\tau), (\delta'(t), \frac{\partial}{\partial t}\mathcal{E}_1(z-1, \tau-t))) - (\varphi(z,\tau), (\delta'(t), \frac{\partial}{\partial t}\mathcal{E}_1(z+1, \tau-t))) = (\varphi(z,\tau), \frac{1}{2}\delta(\tau - |z-1|)) + (\varphi(z,\tau), \frac{1}{2}\delta(\tau-|z+1|)) = (\frac{1}{2}[\delta(t-|x-1|) + \delta(t-|x+1|)], \varphi(x,t))$.

В-дь:

$u(x,t) = \frac{1}{2\alpha}\left\{\exp[\alpha(t-|x|)] - 1\right\}\theta(t-|x|) + \frac{1}{2}[\delta(t-|x-1|) + \delta(t-|x+1|)]$.

N465

$u_t = 16 u_{xx} + t^2 \sin x$

$u(x,0) = x^3 \theta(5-x)$.

Введемо позначення:

$f(x,t) = t^2 \sin x$

$u_0(x,t) = x^3 \theta(5-x)$.

$$\mathcal{E}(x,t) = \frac{\theta(t)}{8\sqrt{\pi t}} e^{-\frac{x^2}{64t}}$$

$$u(x,t) = \int_0^t ds \int_{-\infty}^{+\infty} dy \frac{\theta(s)}{8\sqrt{\pi s}} e^{-\frac{y^2}{64s}} \sin(x-y)(t-s)^2 +$$

$$+ \int_{-\infty}^{+\infty} dy \frac{\theta(t)}{8\sqrt{\pi t}} e^{-y^2/64t} \cdot (x-y)^3 \theta(5-x+y) =$$

$$= \int_0^t ds \frac{\theta(s)(t-s)^2}{8\sqrt{\pi s}} \underbrace{\int_{-\infty}^{+\infty} dy\, e^{-\frac{y^2}{64s}} \sin(x-y)}_{I_1} +$$

$$+ \frac{\theta(t)}{8\sqrt{\pi t}} \underbrace{\int_{-\infty}^{+\infty} dy\, e^{-y^2/64t} \sin(x-y)^3 \theta(5-x+y)}_{I_2} .$$

$$I_1 = \int_{-\infty}^{+\infty} dy\, e^{-y^2/64s} \sin(x-y) = \{\text{Пуассон}\} =$$

$$= 8 e^{-16s}\sqrt{\pi s} \sin x ; \Rightarrow \int_{-\infty}^{+\infty} e^{-\alpha y^2} dy = \sqrt{\frac{\pi}{\alpha}} ;$$

$$I_2 = \int_{-\infty}^{+\infty} dy\, e^{-y^2/64t} (x-y)^3 \theta(5-x+y) = -32 e^{-\frac{(-5+x)^2}{64t}} \cdot$$

$$\cdot t(25+64t+x(5+x)) + 4\sqrt{\pi t}\, x(96t+x^2)\, \mathrm{erfc}\left(\frac{x-5}{8\sqrt{t}}\right)$$

где $\mathrm{erfc} = 1 - \mathrm{erf}$.

$$\int_0^t ds(t-s)^2 8\sqrt{\pi s}\, e^{-16s} \sin x = \sin \int_0^t ds\,(t-s)^2 e^{-16s} =$$

$$= \left(\frac{1 - e^{16t} - 16t + 128t^2}{2048} \right) \sin 4x;$$

$$u(x,t) = \left(\frac{1 - 16t + 128t^2 - e^{-16t}}{2048} \right) \sin 4x +$$

$$+ \frac{x}{2}(56t + x^2)\,\mathrm{erfc}\left(\frac{x-5}{8\sqrt{t}}\right) - 4e^{-\frac{(-5+x)^2}{64t}} \cdot$$

$$\cdot \sqrt{\frac{t}{\pi}}\,(25 + 64t + x(5+x)).$$

N467

$$u_t = \Delta u + 2x_1x_2x_3 \qquad x \in \mathbb{R}^3,\ t > 0$$

$$u(x_1, x_2, x_3, 0)\ x_1^2 + x_2^2 - 2x_3^2$$

$$u(x,t) = \int_0^t ds \int_{\mathbb{R}^3} dy\, \mathcal{E}_3(y,s) f(x-y, t-s) + \int_{\mathbb{R}^3} dy\, \mathcal{E}_3 \cdot$$

$$\cdot (y,t)\, u_0(x,y).$$

$$\begin{cases} \mathcal{E}_3(x,t) = \dfrac{\theta(t)}{(2\sqrt{\pi t})^3} e^{-\frac{|x|^2}{4t}} = \dfrac{\theta(t)}{8(\pi t)^{3/2}} e^{-\frac{x_1^2 + x_2^2 + x_3^2}{4t}} \\ f(x,t) = 2x_1x_2x_3 \\ u_0(x) = x_1^2 + x_2^2 - 2x_3^2 \end{cases}$$

I_1

$$u(x,t) = \int_0^t ds \int_{\mathbb{R}^3} dy \frac{\theta(s)}{8(\pi s)^{3/2}} e^{-\frac{y_1^2 + y_2^2 + y_3^2}{4s}} (2(x_1 -$$

$$- y_1)(x_2 - y_2)(x_3 - y_3)) + \int_{\mathbb{R}^3} dy \frac{\theta(t)}{8(\pi t)^{3/2}} \cdot$$

$$\cdot e^{\frac{-y_1^2+y_2^2+y_3^2}{4t}}\left(\left((x_1-y_1)^2+(x_2-y_2)^2\cdot 2(x_3-y_3)^2\right)\right);$$

$$\to I_2$$

$$I_1(x,s)=\int_0^t ds\int_{R^3} dy\,\frac{\theta(s)}{8(\pi s)^{3/2}}\,e^{\frac{-y_1^2+y_2^2+y_3^2}{4s}}\left(2(x_1-y_1)\cdot\right.$$

$$\left.\cdot(x_2-y_2)(x_3-y_3)\right)=2\int_0^t\frac{1}{8(\pi s)^{3/2}}ds\int_{-\infty}^{+\infty}(x_1-y_1)e^{\frac{-y_1^2}{4s}}dy_1\cdot$$

$$\cdot\int_{-\infty}^{+\infty}(x_2-y_2)e^{\frac{-y_2^2}{4s}}dy_2\int_{-\infty}^{+\infty}(x_3-y_3)\cdot e^{\frac{-y_3^2}{4s}}dy_3=$$

$$=\frac{1}{16\pi^{3/2}}\int_0^t\frac{ds}{s^{3/2}}(2\sqrt{\pi}\sqrt{s})^3x_1x_2x_3=\frac{8\pi^{3/2}}{16\pi^{3/2}}x_1x_2x_3\cdot$$

$$\cdot\int_0^t\frac{s^{3/2}}{s^{3/2}}ds=\frac{x_1x_2x_3}{2}s\Big|_0^t=\frac{x_1x_2x_3t}{2};$$

$$I_2(x,t)=\int_{R^3}dy\,\frac{\theta(t)}{8(\pi t)^{3/2}}\cdot e^{\frac{-y_1^2+y_2^2+y_3^2}{4t}}\left((x_1-y_1)^2+\right.$$

$$\left.+(x_2-y_2)^2\cdot 2(x_3-y_3)^2\right)=e^{\frac{-y_2^2+y_3^2}{4t}}\cdot\frac{1}{8(\pi t)^{3/2}}\cdot$$

$$\cdot\int_{-\infty}^{+\infty}e^{\frac{-y_1^2}{4t}}(x_1-y_1)^2dy_1+\frac{e^{\frac{-y_1^2+y_2^2}{4t}}}{8(\pi t)^{3/2}}\int_{-\infty}^{+\infty}e^{\frac{-y_2^2}{4t}}\cdot$$

$$\cdot(x_2-y_2)^2dy_2-2\int_{-\infty}^{+\infty}e^{\frac{-y_3^2}{4t}}(x_3-y_3)^2dy_3\cdot$$

$$\cdot e^{\frac{-y_1^2+y_2^2}{4t}}\cdot\frac{1}{8(\pi t)^{3/2}}=e^{\frac{-y_2^2+y_3^2}{4t}}\frac{1}{8(\pi t)^{3/2}}\cdot$$

$$\cdot 2\sqrt{\pi t}\cdot(x_1^2+2t)+e^{\frac{-y_1^2+y_2^2}{4t}}\cdot\frac{1}{8(\pi t)^{3/2}}\cdot 2\sqrt{\pi t}\cdot$$

$\cdot (x_2^2+2t) - e^{\frac{-y_1^2+y_2^2}{4t}} \frac{1}{8(\pi t)^{3/2}} 2\sqrt{\pi t} \cdot$

$\cdot (x_3^2+2t) = \frac{2\sqrt{\pi t}}{8(\sqrt{\pi t})^3} \Big(e^{\frac{-y_2^2+y_3^2}{4t}} (x_1^2+2t) +$

$+ e^{\frac{-y_2^2+y_3^2}{4t}} (x_2^2+2t) - e^{\frac{-y_2^2+y_3^2}{4t}} (x_3^2+2t) \Big)$

$$u(x,t) = x_1 x_2 x_3 t + \frac{1}{4\pi t} \Big(e^{\frac{-y_2^2+y_3^2}{4t}} (x_1^2+2t) +$$

$$+ e^{\frac{-y_1^2+y_3^2}{4t}} (x_2^2+2t) - 2e^{\frac{-y_1^2+y_2^2}{4t}} (x_3^2+2t) \Big)$$

N 485

$u_{tt} = u_{xx} + \theta(t)\delta(x)$

$u(x,0) = \delta(x-x_0)$

$u_t(x,0) = x\delta(x)$

$F(x,t) = \theta(t)\theta(t)\delta(x) + \delta'(t)\delta(x-x_0) + \delta(t) \cdot$
$x\delta(x)$;

$u(x,t) = (\varepsilon_n * f)(x,t)$

$\varepsilon = \frac{\theta(t)}{2} \theta(t-|x|)$;

$$u_1 = \varepsilon_n * \theta(t)\theta(t)\delta(x) = \varepsilon_n * \theta(t)\theta(t) =$$

$$= \int_{-\infty}^{+\infty} \varepsilon_1(x,t') \cdot \theta(t-t')\theta(t-t')\, dt' =$$

$$= \int_{-\infty}^{+\infty} \frac{\theta(t')}{2} \theta(t'-|x|)\theta(t-t')\theta(t-t')dt' =$$

$$= \frac{\theta(0)}{2}\int_0^t \theta(t')\theta(t'-|x|)dt' =$$

$$= \left\{ \begin{array}{l} u = \theta(t')\theta(t'-|x|) ; \\ du = \delta(t')\theta(t'-|x|) + \delta(t'-|x|) , \\ d\vartheta = dt' ; \quad \vartheta = t \end{array} \right\} =$$

$$= \frac{1}{4} t' \theta(t')\theta(t'-|x|)\Big|_0^t - \frac{1}{4}\int_0^t \delta(t')\theta(t'-|x|)dt' -$$

$$-\frac{1}{4}\int_0^t \theta(t')\delta(t'-|x|)dt' = \frac{1}{4} t\theta(t)\theta(t-|x|) -$$

$$-\frac{1}{4}\int_0^t \delta(t')\theta(-|x|)dt' - \frac{1}{4}\int_0^t \theta(t'+|x|\delta\cdot$$

$$\cdot(t'-|x|)d(t'-|x|) = \frac{1}{4} t\theta(t)\theta(t-|x|) -$$

$$-\frac{1}{4}\theta(t)\theta(-|x|) + \frac{1}{8}\theta(-|x|) - \frac{1}{4}\theta(|x|)\cdot$$

$$\cdot\theta(t-|x|) + \frac{1}{4}\theta(|x|)\theta(-|x|) = \frac{t}{4}\theta(t)\theta(t) -$$

$$-\frac{t}{4}\theta(t)\theta(|x|) - \frac{1}{4}\theta(t) + \frac{1}{4}\theta(|x|).$$

$$u_2 = \xi_3 * \delta'(t)\delta(x-x_0) = (\xi_3(x+x_0,t) * \delta'(t)) =$$

$$= \frac{\delta(t)}{2}\theta(t-|x+x_0|) + \frac{\theta(t)}{2}\delta(t-|x+x_0|).$$

$$U_3 = \mathcal{E}_3 * \delta(t)\delta(x) = \mathcal{E}_3(x,t) = \frac{\theta(t)}{2} \cdot$$

$$\cdot\, \theta(t-|x|);$$

$$U_3 = \mathcal{E}_3 * x = \int_{-\infty}^{+\infty} \frac{\theta(t)}{2}\theta(t-|x'|)\cdot(x-x')\,dx'.$$

N500

$$\begin{cases} U_{tt} = U_{xx} + e^{-2t}e^{x} \\ U(x,0) = x\,\theta(x) \\ U_t(x,0) = x^2\theta(x) \end{cases}$$

$$U = \frac{1}{2}\int_0^t ds \int_{x-t+s}^{x+t-s} dy\, e^{-2y}e^{-2s} + \mathcal{E}_1 * \delta'(t)\, x\,\theta(x)$$

$$+ \mathcal{E}_1 * \delta(t)\, x^2\theta(x)$$

$$J = \frac{1}{2}\int_0^t e^{-2s}ds \int_{x-t+s}^{x+t-s} e^{-2y}dy = -\int_0^t e^{-2s}e^{-2y}\Big|_{x-t+s}^{x+t-s} =$$

$$= -e^{-2x}e^{-2t}t - \frac{1}{4}e^{-2x}e^{2t}e^{-4s}\Big|_0^t = -e^{-2x}e^{-2t}t -$$

$$-\frac{1}{4}e^{-2x}e^{-2t}e^{-4t} + \frac{1}{4}e^{-2x}e^{2t}.$$

$$\mathcal{E}_1 = \frac{\theta(t)}{2}\theta(t-|x|) \to \tau \Rightarrow$$

$$\Rightarrow \frac{\theta(t)}{2}\theta(t-|x| * \delta'(t)\, x\,\theta(x) = \frac{\delta(t)}{2}\theta(t -$$

$$-|x|) * \delta(t)\, x\theta(x) + \frac{\theta(t)}{2}\delta(t-|x|) * \delta'(t)x \cdot$$

$$\cdot \theta(x) = \frac{\theta(t-|x|)}{2} * x\theta(x) + \frac{\theta(t+|x|)}{2} * x\theta(x) =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \theta(t-|x|)(x-x')\theta(x-x')\,dx' = \frac{1}{2}\theta \cdot$$

$$\cdot (t-|x| \int_0^{x} (x-x')\,dx' = \frac{1}{2}\theta(t-|x|)\left(x^2 - \frac{x'^2}{2}\Big|_0^x\right) =$$

$$= \frac{1}{2}\theta(t-|x|)\frac{x^2}{2} = \frac{x^2}{4}\theta(t-|x|)$$

$$u_3 = \mathcal{E}_1 * x^2\theta(x) = \frac{\theta(t)}{2}\theta(t-|x|) * x^2\theta(x) =$$

$$= \frac{\theta(t)}{2}\int_{-\infty}^{+\infty} \theta(t-|x'|(x-x')^2\theta(x-x')\,dt =$$

$$= \frac{\theta(t)}{2}\theta(t-|x|)\int_0^x (x^2 - 2xx' + x'^2)\,dx' =$$

$$= \frac{\theta(t)}{2}\theta(t-|x|)\left(x^3 - 2x\frac{x'^2}{2}\Big|_0^x - \frac{x'^3}{3}\Big|_0^x\right) =$$

$$= \frac{\theta(t)}{2}\theta(t-|x|)\left(x^3 - x^3 + \frac{x^3}{3}\right) = \frac{x^3}{6}\theta(t) \cdot$$

$$\cdot \theta(t-|x|)$$

$$u(x,0) = -\frac{1}{4}e^{-2x} + \frac{1}{4}e^{-2x} + \frac{x^2}{4}\theta(-|x|) +$$

$$+ \frac{x^3}{12}\theta(-|x|).$$