Домашняя работа №5

№246

$$\begin{cases} \Delta u = 0 \\ u(R,\theta,\varphi) = \sin\theta(\sin\varphi + \sin\theta) \end{cases}$$

$\sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2i}$;

$$\sin\theta(\sin\varphi + \sin\theta) = \sin\theta\sin\varphi + \sin^2\theta =$$
$$= \frac{e^{i\varphi}}{2i}\sin\theta - \frac{e^{-i\varphi}}{2i}\sin\theta + \sin^2\theta = \sqrt{\frac{8\pi}{3}} \cdot$$
$$\cdot \frac{Y_{1,1}}{2i} + \sqrt{\frac{8\pi}{3}} \cdot \frac{Y_{1,-1}}{2i} + 1 - \cos^2\theta =$$
$$= \Big[Y_{2,0} = \sqrt{\frac{5}{16\pi}} \cdot (3\cos^2\theta - 1) = 3\sqrt{\frac{5}{16\pi}}\cos^2\theta -$$
$$- \sqrt{\frac{5}{16\pi}}\cos^2\theta - 1 = (3\cos^2\theta - 1) - 2 = Y_{2,0} \cdot$$
$$\cdot \sqrt{\frac{16\pi}{5}} - Y_{0,0} \cdot 2 \cdot \sqrt{4\pi} \Big] = \sqrt{\frac{8\pi}{3}} \cdot \frac{1}{2i}\left(Y_{1,1} + Y_{1,-1}\right) -$$
$$- Y_{2,0}\sqrt{\frac{16\pi}{5}} + Y_{0,0} \cdot 2\sqrt{4\pi};$$

$$u^{int}(r,\theta,\varphi) = \sum_{l,m} C_{l,m} Y_{l,m} = \sqrt{\frac{8\pi}{3}} \cdot \frac{1}{2i}\Big(Y_{1,1} +$$
$$+ Y_{1,-1}\Big) - Y_{2,0} \cdot \sqrt{\frac{16\pi}{5}} + Y_{0,0} \cdot 2\sqrt{4\pi} \Rightarrow$$

$\Rightarrow C_{0,0} = 2\sqrt{4\pi}$;

$C_{2,0} = \sqrt{\dfrac{16\pi}{5}}$; $\quad C_{1,-1} = \sqrt{\dfrac{8\pi}{3}} \Big/ 2i$.

$$U^{int}(r,\theta,\varphi)=\left(\frac{r}{R}\right)^0\cdot 2\sqrt{4\pi}\,Y_{0,0}+\frac{r}{R}\sqrt{\frac{8\pi}{3}}\cdot$$

$$\cdot\frac{1}{2i}\,Y_{1,1}+\frac{r}{R}\,\frac{\sqrt{\frac{8\pi}{3}}}{2i}\,Y_{1,-1}-\left(\frac{r}{R}\right)^2\sqrt{\frac{16\pi}{5}}=$$

$$=2\sqrt{4\pi}\,Y_{0,0}+\frac{r}{R}\sqrt{\frac{8\pi/3}{2i}}\left(Y_{1,1}+Y_{1,-1}\right)-$$

$$-\left(\frac{r}{R}\right)^2\sqrt{\frac{16\pi}{5}};$$

$$U^{ext}(r,\theta,\varphi)=\frac{R}{r}\cdot 2\sqrt{4\pi}\,Y_{0,0}+\left(\frac{R}{r}\right)^2\cdot$$

$$\cdot\sqrt{\frac{8\pi/3}{2i}}\left(Y_{1,1}+Y_{1,-1}\right)-\left(\frac{R}{r}\right)^3\sqrt{\frac{16\pi}{3}}.$$

№248

$$\left\|\begin{array}{l}\Delta U=0\\ U(R,\theta,\varphi)-U_r(R,\theta,\varphi)=\sin^2\theta\end{array}\right.$$

$$U^{int}(r,\theta,\varphi)=\sum_{l,m}C_{l,m}\left(\frac{r}{R}\right)^l Y_{l,m};$$

$$U_r^{int}(r,\theta,\varphi)=\sum_{l,m}l\,C_{l,m}\frac{r^{l-1}}{R^l}\,Y_{l,m};$$

$$U^{ext}(r,\theta,\varphi)=\sum_{l,m}C_{l,m}\left(\frac{R}{r}\right)^{l+1}Y_{l,m};$$

$$U_r^{ext}(r,\theta,\varphi)=\sum_{l,m}-(l+1)\,C_{l,m}\frac{R^{l+1}}{r^{l+2}}\,Y_{l,m};$$

$$\sin^2\theta=1-\cos^2\theta=Y_{0,0}\cdot 2\sqrt{4\pi}-\sqrt{\frac{16\pi}{5}}\,Y_{2,0};$$

$$U^{int}-U_r^{int}=\sum_{l,m}C_{l,m}\left(\frac{r}{R}\right)^l Y_{l,m}\left(1+\frac{l}{r}\right);$$

$$u^{int}(R,\theta,\varphi) - u_r^{int}(R,\theta,\varphi) = \sum_{l,m} C_{l,m} Y_{l,m} \cdot$$
$$\cdot\left(1+\frac{l}{R}\right) = Y_{0,0}\cdot 2\sqrt{4\pi} - \sqrt{\frac{16\pi}{5}}\, Y_{2,0};$$
$$C_{0,0} = 2\sqrt{4\pi};\quad C_{2,0}\cdot\left(1+\frac{2}{R}\right) = -\sqrt{\frac{16\pi}{5}} \Rightarrow$$
$$\Rightarrow C_{2,0} = -\frac{R\sqrt{\frac{16\pi}{5}}}{2+R};$$
$$u^{int}(r,\theta,\varphi) = 2\sqrt{4\pi}\, Y_{0,0} - \left(\frac{r}{R}\right)^2 \frac{R\sqrt{\frac{16\pi}{5}}}{2+R}\, Y_{2,0} =$$
$$= 2\sqrt{4\pi}\, Y_{0,0} - \frac{r^2}{R(2+R)}\cdot Y_{2,0};$$
$$u^{ext} - u_r^{ext} = \sum_{l,m} C_{l,m} Y_{l,m}\left(\frac{R}{r}\right)^{l+1}\left(1+\frac{l+1}{r}\right);$$
$$u^{ext}(R,\theta,\varphi) - u_r^{ext}(R,\theta,\varphi) = \sum_{l,m} C_{l,m} Y_{l,m}\cdot$$
$$\cdot\left(1+\frac{l+1}{R}\right) = Y_{0,0}\, 2\sqrt{4\pi} - \sqrt{\frac{16\pi}{5}}\cdot Y_{2,0} \Rightarrow$$
$$\Rightarrow C_{0,0} = \frac{R\cdot 2\sqrt{4\pi}}{R+1};$$
$$C_{2,0} = \frac{R\sqrt{\frac{16\pi}{5}}}{R+3}$$
$$u^{ext}(r,\theta,\varphi) = \frac{R^2}{r}\cdot\frac{2\sqrt{4\pi}}{R+1}\, Y_{0,0} - \frac{R^3}{r(R+3)}\sqrt{\frac{16\pi}{5}}\, Y_{2,0}.$$

N250

$$u(1,\theta,\varphi) = \cos^2\theta;$$
$$u(2,\theta,\varphi) = \frac{1}{8}(\cos^2\theta + 1) \qquad R_1 = 1;\ R_2 = 2$$
$$u(1,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}(A_{l,m} + B_{l,m})\, Y_{l,m}(\theta,\varphi) =$$
$$= \cos^2\theta;$$

$$u(2,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum (A_{\ell m} + B_{\ell m}) Y_{\ell m}(\theta,\varphi) =$$

$$= \frac{1}{8}(1+\cos^2\theta) = \frac{1}{8} + \frac{1}{24}\left(\sqrt{\frac{16\pi}{5}}\, Y_{1,0+1}\right) =$$

$$= \frac{1}{6} + \frac{1}{24}\sqrt{\frac{16\pi}{5}}\, Y_{1,0} = \frac{1}{6}\sqrt{4\pi}\, Y_{0,0} + \frac{1}{24} \cdot$$

$$\cdot \sqrt{\frac{16\pi}{5}}\, Y_{1,0}.$$

$$A_{2,0} + B_{2,0} = \frac{1}{3}\sqrt{\frac{16\pi}{5}};$$

$$A_{0,0} + B_{0,0} = \frac{1}{3}\sqrt{4\pi}$$

$$A_0 + B_{0,0} \cdot \frac{1}{2} = \frac{1}{6}\sqrt{4\pi}$$

$$A_{2,0} \cdot 4 + B_{2,0} \cdot \frac{1}{8} = \frac{1}{14}\sqrt{\frac{16\pi}{5}};$$

$$\begin{cases} 2A_{0,0} + B_{0,0} = \frac{1}{3}\sqrt{4\pi}; \\ A_{0,0} + B_{0,0} = 1/3\sqrt{4\pi}; \end{cases} \quad A_{0,0} = 0, \quad B_{0,0} = \frac{1}{3}\sqrt{4\pi}$$

$$\begin{cases} A_{2,0} + B_{2,0} = \frac{1}{3}\sqrt{\frac{16\pi}{5}} \\ 32A_{2,0} + B_{2,0} = \frac{1}{3}\sqrt{\frac{16\pi}{3}} \end{cases} \quad A_{2,0} = 0, \quad B_{2,0} = \frac{1}{3}\sqrt{\frac{16\pi}{3}}$$

$$u(r,\theta,\varphi) = \frac{1}{8}\sqrt{4\pi} \cdot Y_{0,0}\, r^{-1} \cdot \frac{1}{2}\sqrt{\frac{16\pi}{5}}\, r^{-3} \cdot Y_{2,0}$$

№266

$$V_0 = -E_0 z = -E_0 r P_1(\cos\theta) \quad [P_1(x) = x]$$

$V = V_1 = A r P_1(\cos\theta)$, при $r \le a$

$V = V_2 = \left(Br + \frac{C}{r^2}\right) P_1(\cos\theta)$, всередині

$V = V_3 = \left(-E_0 r + A/r^2\right) P_1(\cos\theta)$, $r \ge b$;

$$A = -\frac{3E_0 K}{\Delta},$$

$$B = -\frac{3(2K+1)}{\Delta} E_0;$$

$$C = -\frac{3(K-1)E_0}{\Delta} a^3;$$

$$D = E_0 b^3 + \left[1 + \frac{(2K+1)b^3}{(K-1)a^3}\right];$$

$$K = \frac{\varepsilon_2}{\varepsilon_1}; \quad \Delta = 9K - 2(K-1)^2\left[\left(\frac{a}{b}\right)^3 - 1\right].$$